PERGAMON

International Journal of Solids and Structures 36 (1999) 185–218

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Some extremum properties of finite-step solutions in elastoplasticity

Anna Pandolfi[a], Angelo Carini[b,*]

[a]*Department of Structural Engineering, Politecnico di Milano, Piazza Leonardo Da Vinci, 32, 20133 Milano, Italy*
[b]*Department of Civil Engineering, University of Brescia, via Branze, 38, 25123 Brescia, Italy*

Received 18 April 1996; in revised form 15 September 1997

Abstract

For the nonholonomic elastic–plastic problem under a given external action history over a time interval, an extremal formulation is given in terms of the complete solution over the whole interval. The assumed elastic–plastic behaviour is of the associated type with piecewise linearized yield surface and linear hardening.

When the loading history is reduced to an infinitesimal increment of the external actions (incremental problem) or when the material behaviour is assumed to be of the holonomic type (finite holonomic step) problem, the functional of the extremal formulation may be split into the sum of two other simpler functionals (previously introduced) whose minimum, for both of them, gives the problem solution under less constraints than in the original problem.

For general non-holonomic loading histories the above splitting is shown to be still possible when a particular change of the complementarity condition of the constitutive law is considered, which leads to a new class of holonomic problems.

It is shown that some problems of this new class, together with a suitable time discretization, represent the schematization of the original problem corresponding to well known numerical integration schemes.

1. Introduction

All the classical extremum principles of the theory of plasticity refer mainly to two formulations of the actual elastic–plastic initial/boundary value problem. The first one is referred to as the incremental problem, aiming at the solution of a continuum problem for infinitesimal increments of the external actions. The second one is generally referred to as a finite holonomic problem dealing with the elastic–plastic response to finite increments of the external actions under the path-independence (holonomy) assumption for the constitutive law (deformation theory of plasticity).

* Corresponding author.

PII: S0020-7683(97)00280-1

The solution of the elastic–plastic initial/boundary value problem, under a given history of external actions, requires following the evolution of the body response. This occurs because the constitutive elastic–plastic law is intrinsically path-dependent or non-holonomic. This response, in terms of displacements, strains, stresses and plastic multipliers, must be evaluated in every point of the body and for every time instant within an assigned time interval, in which the external actions are prescribed; from a numerical point of view this requires correspondingly a space and time discretization.

The time discretization is usually based on a preliminary sub-division of the external action history into a sequence of loading conditions at prescribed time instants. These are normally chosen in such a way that the external action variation over each time interval is not far from linear and such that, inside the step, the behaviour may be considered as path-independent (or holonomic) with an expected good approximation. Examples of time-integration strategies are: the multistage method with piecewise linear yield functions (De Donato and Maier, 1972 and 1973), the forward Euler scheme, the generalized trapezoidal rule, the generalized mid-point rule and the backward-difference method (see e.g. Ortiz and Popov, 1985; Simo and Taylor, 1985; Simo et al., 1988).

It is worth noting that a time integration scheme corresponds to a particular approximation of the original non-holonomic material behaviour by means of a holonomic model. In this way the solution of the original problem is transformed into the solution of a sequence of holonomic problems, for each of which, however, the existence of the variational formulation is no longer guaranteed so that the evaluation of some important properties (such as stability, convergence and accuracy of the solution) may be difficult.

Therefore, many attempts were made by different authors (see e.g. Maier, 1969a; Ponter and Martin, 1972; Franchi and Genna, 1984; Martin, 1989; Borino et al., 1989) in order to give a rational energetic base to the time integration methods. Ponter and Martin (1972), proposed the concept of minimum energy strain path, equivalent to a constitutive model in which plastic strain rates follow a linear path in strain space between two given strain points, a path minimizing plastic dissipation with respect to other paths joining the given strain points (Reddy et al., 1986). In this way, the backward difference method has been shown to possess variational consistency (Martin, 1989). Franchi and Genna (1984) were the first to identify the minimum principle whose solution corresponded to the solution given by a numerical method (Initial Stress). A more general point of view is introduced by Borino et al. (1989): in fact they suggest a consistent time discretization procedure, based on the maximum intrinsic dissipation theorem, in order to transform the original initial/boundary value problem into a set of boundary value problems (one for every step) of holonomic plasticity.

In this context, the present paper proposes and develops an a priori approximation of the material behaviour envisaged in the original non-holonomic finite step boundary value problem, with the following advantages (with respect to the above approach where the material behaviour of the approximate solution was dependent from the time integration scheme adopted): (a) an a priori direct control over the approximation of the material behaviour in the original non-holonomic problem; (b) the existence of a variational formulation of the approximate problem as proved in Section 4; (c) the possibility, by virtue of statement (b), to easily evaluate the stability, the convergence and the accuracy of the numerical solution procedure.

The above mentioned a priori direct control of the material behaviour is achieved using a particular change, through a suitable weight function, of the complementarity conditions of the elastic–plastic constitutive law. Section 4 shows that the functional to be minimized for finding the solution can be split into the sum of simpler functionals, the minima of which characterize the approximate holonomic elastic–plastic solution. In particular, the changes of the complementarity condition of the constitutive law are shown which correspond to well known classical time integration schemes of the original non-holonomic problem.

For the sake of simplicity, the elastic–plastic piecewise linearized constitutive laws (Maier, 1970a; De Donato, 1974; Hodge, 1976) will be referred to.

2. Finite-step problem formulation

Consider an elastic–plastic solid which occupies a volume Ω with the smooth boundary $\Gamma = \Gamma_u \bigcup \Gamma_p$, Γ_u and Γ_p denoting the parts of the surface where displacements and surface tractions are prescribed, respectively. A triaxial orthogonal Cartesian reference system x_i $(i = 1, 2, 3)$ is adopted. Volume forces $F_i^0 + F_i(t)$ and prescribed strains $\theta_{ij}^0 + \theta_{ij}(t)$ in Ω, prescribed displacements $w_i(t)$ on Γ_u and surface forces $p_i^0 + p_i(t)$ on Γ_p, are given for any instant $0 \leqslant t \leqslant T$ through known time functions where F_i^0, θ_{ij}^0, p_i^0 are the external actions at time $t = 0$ and $F_i(t)$, $\theta_{ij}(t)$, $p_i(t)$ are known finite increments of the external actions at time t. Denoting with $u_i(t)$ and $\varepsilon_{ij}(t)$ the displacement and strain fields, respectively, let us assume the initial configuration at time $t = 0$ as reference configuration, therefore $u_i(t = 0) = 0$, $\theta_{ij}^0 = 0$ and $\varepsilon_{ij}(t = 0) = 0$. Moreover we denote the stresses as $\sigma_{ij}^0 + \bar{\sigma}_{ij}(t)$ and the plastic multipliers as $\lambda_\alpha^0 + \bar{\lambda}_\alpha(t)$ where σ_{ij}^0 and λ_α^0 are the (assumed known) stresses and plastic multipliers at time $t = 0$, respectively, while $\bar{\sigma}_{ij}(t)$ and $\bar{\lambda}_\alpha(t)$ represent the unknown finite increments of stresses and plastic multipliers at time t, respectively; F_i^0 and p_i^0 are assumed to be in equilibrium with the stresses σ_{ij}^0.

The incremental stress, strain and displacement space-time functions $\bar{\sigma}_{ij}(x_k, t)$, $\bar{\varepsilon}_{ij}(x_k, t)$ and $\bar{u}_i(x_k, t)$ are to be determined in the volume Ω and in the time interval $\Delta T = [0, T]$.

The assumption is made of small displacements and of elastic–plastic behaviour described by piecewise linearized yield surface with associated flow rule and linear hardening (Maier, 1970a; De Donato, 1974; Hodge, 1976) (see Fig. 1). Then, denoting with n_i the unit outward normal vector to Γ, the set of all the governing equations (equilibrium, compatibility, constitutive law) of the problem, denoted in the following as finite-step problem $\Delta\bar{P}$, reads (the repeated indices summation convention is adopted, except for Greek indices):

Problem $\Delta\bar{P}$ (2.1)–(2.11)

$$\bar{\sigma}_{ij,j} + F_i = 0 \quad \text{in } \Omega \times \Delta T \tag{2.1}$$

$$\bar{\sigma}_{ij} n_j = p_i \quad \text{on } \Gamma_p \times \Delta T \tag{2.2}$$

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) \quad \text{in } \Omega \times \Delta T \tag{2.3}$$

$$\bar{u}_i = w_i \quad \text{on } \Gamma_u \times \Delta T \tag{2.4}$$

$$\bar{\sigma}_{ij} = D_{ijhk} \bar{\varepsilon}_{hk}^e \quad (\text{or } \bar{\varepsilon}_{ij}^e = C_{ijhk} \bar{\sigma}_{hk}) \tag{2.5}$$

$$\bar{\varepsilon}_{ij}^e = \bar{\varepsilon}_{ij} - \bar{\varepsilon}_{ij}^p - \theta_{ij} \tag{2.6}$$

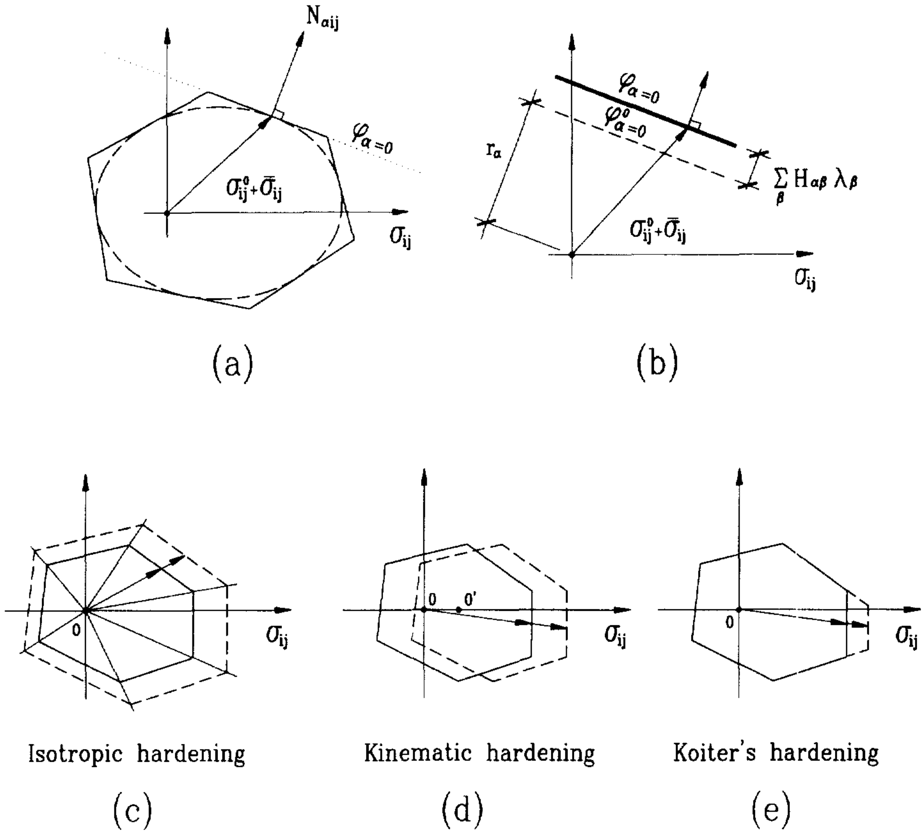


Fig. 1. (a)–(b) The piecewise linearization of the yield surface with α ($\alpha = 1, \ldots, m$) planes being $N_{\alpha ij}$ the relevant outward normal unit vectors and r_α the distance of each plane from the origin at the virgin state. (c)–(e) Some common hardening yield surface evolutions depending on the choice of the yield plane interaction matrix $H_{\alpha\beta}$ (Maier, 1970a).

$$\bar{\phi}_\alpha = \phi_\alpha(\bar{\sigma}_{ij}, \bar{\lambda}_\alpha) = N_{\alpha ij}(\bar{\sigma}_{ij} + \sigma^0_{ij}) - \sum_\beta H_{\alpha\beta}(\bar{\lambda}_\beta + \lambda^0_\beta) - r_\alpha$$

$$= N_{\alpha ij}\bar{\sigma}_{ij} - \sum_\beta H_{\alpha\beta}\bar{\lambda}_\beta + \phi^0_\alpha \quad (\alpha, \beta = 1, \ldots, m) \tag{2.7}$$

$$\dot{\bar{\varepsilon}}^p_{ij} = \sum_\alpha \frac{\partial \bar{\phi}_\alpha}{\partial \bar{\sigma}_{ij}} \dot{\bar{\lambda}}_\alpha = \sum_\alpha N_{\alpha ij} \dot{\bar{\lambda}}_\alpha \tag{2.8}$$

$$\bar{\phi}_\alpha \leqslant 0 \tag{2.9}$$

$$\dot{\bar{\lambda}}_\alpha \geqslant 0 \tag{2.10}$$

$$\bar{\phi}_\alpha \dot{\bar{\lambda}}_\alpha = 0 \tag{2.11}$$

where a superimposed dot means derivative with respect to time t.

Equations (2.1)–(2.2) and (2.3)–(2.4) express the incremental equilibrium and compatibility, respectively. Equation (2.5) defines the elastic part of the constitutive law, $D_{ijhk} = C^{-1}_{ijhk}$ being the elastic moduli tensor. D_{ijhk} is assumed with the usual properties of symmetry and positive definiteness. Equation (2.6) states the additivity of the elastic $\bar{\varepsilon}^{e}_{ij}$, plastic $\bar{\varepsilon}^{p}_{ij}$ and inelastic (prescribed) θ_{ij} strains. Equation (2.7) defines the piecewise linearized yield surface in the stress space, m being the number of planes $\bar{\phi}_{\alpha} = 0$, $N_{\alpha ij}$ the relevant outward normal unit vectors, r_{α} the distance of each plane from the origin at the material virgin state. $H_{\alpha\beta}$ is the yield plane interaction matrix describing the hardening law, $\bar{\lambda}_{\alpha}$ represents the plastic multipliers and $\phi^0_{\alpha} = N_{\alpha ij}\sigma^0_{ij} - \Sigma_{\beta} H_{\alpha\beta}\lambda^0_{\beta} - r_{\alpha}$ the value of $\bar{\phi}_{\alpha}$ at the initial time $t = 0$. In the following $N_{\alpha ij}$, r_{α} and $H_{\alpha\beta}$ are assumed to be constant with respect to the time variable t, and, besides, $H_{\alpha\beta}$ is assumed positive semidefinite. Equation (2.8) expresses the associated flow law (normality) while eqns (2.9)–(2.11) express the loading–unloading criterion (Prager's consistency rule); eqn (2.11) is often called complementarity rule.

Using the above piecewise linearization, any yield surface can be easily approximated with the requested degree of accuracy; by a suitable choice of the $H_{\alpha\beta}$ yield plane interaction matrix, the most common hardening yield surface evolutions can be represented (see e.g. Maier, 1970a; De Donato, 1974; Hodge, 1976) (see Fig. 1).

In the following the time interval ΔT is always conceived as a time step within an assigned bigger time interval $\mathscr{T} = [T_0, T_f]$ where the solution is sought. The problem of determining the elastic–plastic response in the whole interval $\mathscr{T}$, will be denoted as problem $\bar{P}$. Throughout the paper we will mainly consider the finite-step problem $\Delta\bar{P}$ and its approximation. Only in Sections 3.3 and 6 the elastic–plastic problem $\bar{P}$, in the whole time interval $\mathscr{T}$, will be considered. For the sake of simplicity, the lower and upper limits of the time interval ΔT will be taken (as already assumed above) as 0 and T, respectively, which it is always possible to lead back to with a suitable change of time variable.

3. Approximate time integration of the initial/boundary value problem

3.1. *Remarks on time discretization and problem reformulation*

Usually the whole time interval $\mathscr{T}$, where the solution is sought, is subdivided into a given number of pre-defined subintervals in each of which, whatever numerical integration algorithm is used, the step problem always amounts to a deformation theory or holonomic plasticity problem (Borino et al., 1989).

From the numerical point of view this suggests making an a priori approximation of the original non-holonomic finite-step problem $\Delta\bar{P}$ (eqns (2.1)–(2.11)) by other ones of the holonomic type, which will be simpler, easier to handle (see Section 4) and able to allow for the time discretization of the problem with a Ritz-type technique (see Section 5). Denoting with unbarred symbols the variables relevant to the solution of these new finite-step holonomic type problems, their formulation may be easily obtained by substituting the complementarity condition eqn (2.11) with the following one:

$$\phi_{\alpha}(t)\int_0^T S(t,\tau)\lambda_{\alpha}(\tau)\,d\tau = 0 \quad \forall t \in \Delta T \tag{3.1}$$

where, among all possible weight functions $S(t,\tau)$, only those satisfying the following relations are considered:

$$-\frac{\partial S(t,\tau)}{\partial \tau} \equiv R(t,\tau) = R(\tau,t) \tag{3.2}$$

$$S(t,\tau) \geqslant 0 \quad \forall t,\tau \in \Delta T; \qquad \int_0^T \int_0^T S(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}t > 0 \tag{3.3}$$

$$\int_0^T \int_0^T R(t,\tau)v(t)v(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \geqslant 0 \quad \forall v(t) \not\equiv 0 \tag{3.4}$$

$$S(t,T) \equiv 0 \tag{3.5}$$

being $v(t)$ any time function for which the integral of relation eqn (3.4) makes sense.

Then the set of the governing equations of the above finite-step holonomic type problem (Problem ΔP) becomes:

Problem ΔP (3.6)–(3.16)

$$\sigma_{ij,j} + F_i = 0 \quad \text{in } \Omega \times \Delta T \tag{3.6}$$

$$\sigma_{ij} n_j = p_i \quad \text{on } \Gamma_p \times \Delta T \tag{3.7}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } \Omega \times \Delta T \tag{3.8}$$

$$u_i = w_i \quad \text{on } \Gamma_u \times \Delta T \tag{3.9}$$

$$\sigma_{ij} = D_{ijhk}\varepsilon^e_{hk} \quad (\text{or } \varepsilon^e_{ij} = C_{ijhk}\sigma_{hk}) \tag{3.10}$$

$$\varepsilon^e_{ij} = \varepsilon_{ij} - \varepsilon^p_{ij} - \theta_{ij} \tag{3.11}$$

$$\begin{aligned}\phi_\alpha = \phi_\alpha(\sigma_{ij}, \lambda_\alpha) &= N_{\alpha ij}(\sigma_{ij} + \sigma^0_{ij}) - \sum_\beta H_{\alpha\beta}(\lambda_\beta + \lambda^0_\beta) - r_\alpha \\ &= N_{\alpha ij}\sigma_{ij} - \sum_\beta H_{\alpha\beta}\lambda_\beta + \phi^0_\alpha \quad (\alpha, \beta = 1, \ldots, m)\end{aligned} \tag{3.12}$$

$$\dot{\varepsilon}^p_{ij} = \sum_\alpha \frac{\partial \phi_\alpha}{\partial \sigma_{ij}} \dot{\lambda}_\alpha = \sum_\alpha N_{\alpha ij}\dot{\lambda}_\alpha \tag{3.13}$$

$$\phi_\alpha \leqslant 0 \tag{3.14}$$

$$\dot{\lambda}_\alpha \geqslant 0 \tag{3.15}$$

$$\phi_\alpha(t) \int_0^T S(t,\tau)\dot{\lambda}_\alpha(\tau)\,\mathrm{d}\tau = 0 \quad \forall t \in \Delta T. \tag{3.16}$$

The choice of the weight function $S(t,\tau)$ and the relevant constraints (3.1)–(3.5) will be seen (Sections 3.3, 3.4 and 4) to play a fundamental role on the accuracy, on the contractivity and on the extremal properties of the solution of the approximate problem ΔP and seem to strengthen the interest for the kind of approximation here introduced.

3.2. *The choice of weight function* S: *some examples*

An infinite number of functions exists which satisfy the conditions (3.2)–(3.5). Examples of such functions below.

Example 1 [see Fig. 2(a) and 2(b)].

$$S_1(t,\tau) = \frac{1}{t}\left[\exp\left(\frac{t}{T}\right) - \exp\left(\frac{t\tau}{T^2}\right)\right]. \tag{3.17}$$

The complementarity condition, by integration by part, becomes:

$$\phi_\alpha(t)\int_0^T \exp\left(\frac{t\tau}{T^2}\right)\lambda_\alpha(\tau)\,\mathrm{d}\tau = 0. \tag{3.18}$$

Example 2 [see Fig. 2(c) and 2(d)].

$$S_2(t,\tau) = \left(1-\frac{\tau}{T}\right)\exp\left(1-\frac{t}{T}-\frac{\tau}{T}+\frac{t\tau}{T^2}\right). \tag{3.19}$$

For this example the complementarity condition reads:

$$\phi_\alpha(t)\int_0^T \left[1+\left(1-\frac{t}{T}\right)\left(1-\frac{\tau}{T}\right)\right]\exp\left(1-\frac{t}{T}-\frac{\tau}{T}+\frac{t\tau}{T^2}\right)\lambda_\alpha(\tau)\,\mathrm{d}\tau = 0. \tag{3.20}$$

Example 3 [see Fig. 2(e) and 2(f)].

$$S_3(t,\tau) = \beta(t)\gamma(t,\tau) \tag{3.21}$$

where

$$\beta(t) = \exp\left(-\frac{T^2}{4t(T-t)}\right) \tag{3.22}$$

$$\gamma(t,\tau) = \int_\tau^T \exp\left(\frac{t\eta}{T^2}\right)\beta(\eta)\,\mathrm{d}\eta. \tag{3.23}$$

In this case the complementarity condition becomes:

$$\phi_\alpha(t)\beta(t)\int_0^T \exp\left(\frac{t\tau}{T^2}\right)\beta(\tau)\lambda_\alpha(\tau)\,\mathrm{d}\tau = 0. \tag{3.24}$$

It is worth noting that the substitution of the complementarity conditions eqn (2.11) with eqn (3.1) is equivalent to the substitution of eqn (2.11) with:

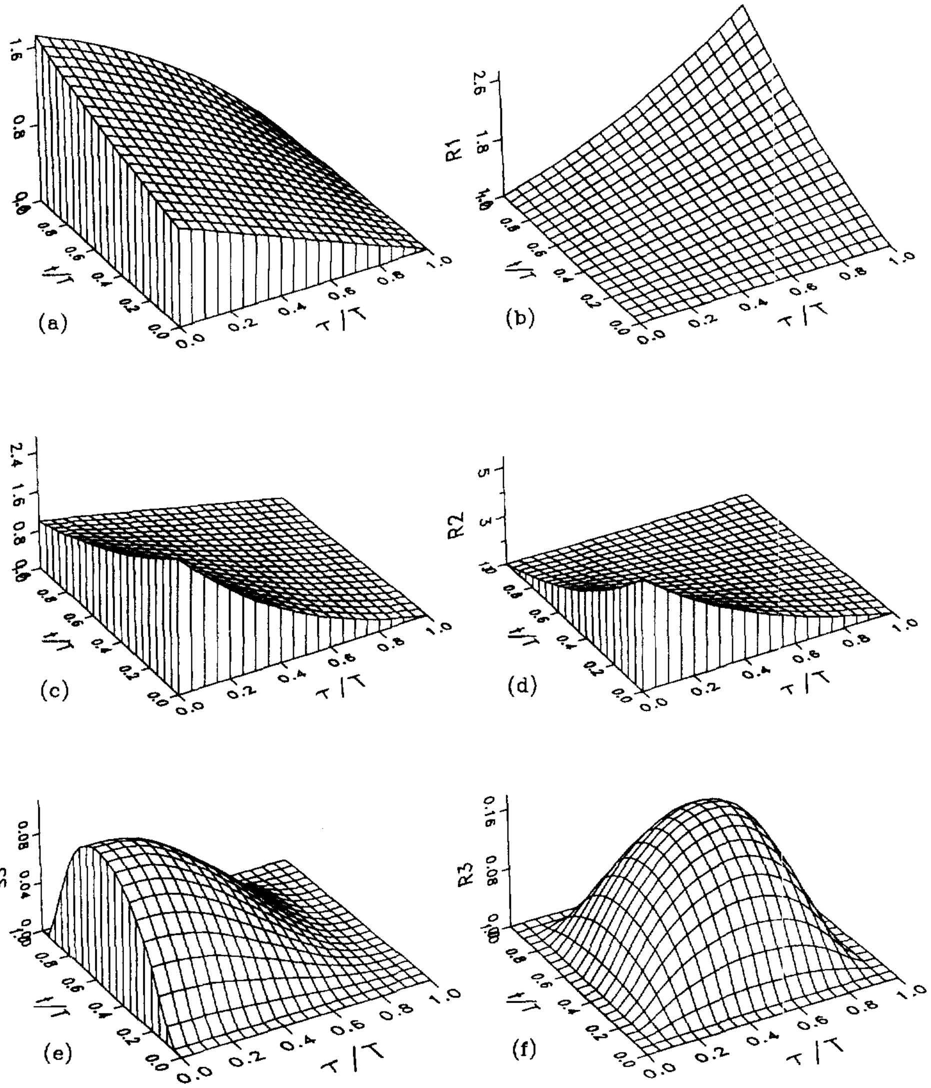


Fig. 2. On the choice of the weight function $S(t, \tau)$. Plot of the functions $S_1(t, \tau)$, $S_2(t, \tau)$, $S_3(t, \tau)$ and their derivatives $R_1(t, \tau) = R_1(\tau, t)$, $R_2(t, \tau) = R_2(\tau, t)$, $R_3(t, \tau) = R_3(\tau, t)$ of examples 1 [eqn (3.17)], 2 [eqn (3.19)] and 3 [eqn (3.21)], respectively.

$$\phi_\alpha \dot{\lambda}^s_\alpha = 0 \tag{3.25}$$

where

$$\dot{\lambda}^s_\alpha = \int_0^T S(t, \tau) \dot{\lambda}_\alpha(\tau)\, \mathrm{d}\tau \tag{3.26}$$

represents a sort of weighted average of the function λ_α over the time interval [0, T]. Taking into account eqns (3.2) and (3.5) and integrating by part the right-hand side of eqn (3.26) we can write:

$$\dot{\lambda}_\alpha^s = \int_0^T R(t,\tau)\lambda_\alpha(\tau)\,d\tau. \tag{3.27}$$

This shows the role of the symmetric function $R(t,\tau) = R(\tau,t)$. The plots of the $R(t,\tau)$ functions of Fig. 2, visualize the different weights given to the values of λ_α in the interval [0, T] [in Fig. 2(b) larger weights are given to λ for $t \to T$ while vanishing weights are given for $t \to 0$; in Fig. 2(d) larger weights are given for $t \to 0$ and vanishing weights for $t \to T$; finally in Fig. 2(f) the larger values are given around $t = T/2$].

3.3. *Accuracy analysis for the approximate evolution problem*

Suppose that (a) the solutions of the original and of the approximate problems exist and are unique in the time interval ΔT and (b) the solution of the original problem is of holonomic type, that is $\dot{\lambda}_\alpha$ is either $\equiv 0$ or >0 over all the interval ΔT. Under these hypotheses it is possible to prove that the exact and the approximate solutions relevant to the same interval ΔT coincide.

To this aim let us compare the new approximate problem ΔP, eqns (3.6)–(3.16), with the original one $\Delta \bar{P}$, eqns (2.1)–(2.11), under the above hypotheses (a) and (b). The following can be said:

1. the two problems are different only for the complementarity conditions, eqns (2.11) and (3.16);
2. as a consequence of condition eqn (3.15), it follows that over all the same subinterval either $\int_0^T S(t,\tau)\dot{\lambda}_\alpha\,d\tau \equiv 0$ or >0;
3. from the preceding point 2 and from the hypotheses (a) and (b) it follows that the solutions of the original problem $\Delta\bar{P}$ and of the approximate problem ΔP coincide. In fact the approximate complementarity condition, eqn (3.16), is satisfied in the time interval ΔT by the solution of the original problem $\Delta\bar{P}$, and so all of the remaining equations of the approximate problem ΔP are satisfied (coinciding with the corresponding equations of the original problem). Therefore, owing to the assumed uniqueness of solution of both original and approximate problem, the solution of the original problem $\Delta\bar{P}$ is also the solution of the approximate problem ΔP in the time interval ΔT.

It is possible to show that for every loading history continuous in time and in the presence of a space discretization (for instance by the application of the Finite Element Method) the solution of the original problem $\bar{P}$ (over the whole time interval $\mathscr{T}$) is amenable, in an exact way, to the solution of a finite sequence of holonomic problems $\Delta\bar{P}$ each of which are relevant to time subintervals defined by a finite number of distinct (a priori unknown) time instants $T_0 < T_1^* < T_2^* < \cdots < T_n^* < T_f$ (see De Donato and Maier, 1972 and 1973).

In the case of an a priori knowledge of the above time instants T_i^* ($i = 1,\ldots,n$) it becomes obvious that the set of approximate solutions for each subinterval coincides with the exact solution relevant to the whole time interval $\mathscr{T}$.

Of course, in the absence of an a priori knowledge of the above finite distinct instants T_i^* ($i = 1,\ldots,n$), the usual subdivision of the time interval $\mathscr{T} = [T_0, T_f]$ in a given number (for instance equal) of subintervals, can lead to the coincidence of the exact and approximate unknown functions (with all their derivatives) only when the number of subintervals tend to infinity uniformly.

Lastly, it is worth noting that, in the absence of a space discretization, it becomes far more complex to make an accuracy analysis for every given function $S(t, \tau)$ as defined in Subsection 3.1. In this context, studies of the authors are in progress on the existence of particular weight functions $S(t, \tau)$ for which an a priori given degree of accuracy is assured.

3.4. *Contractivity property of the approximate evolution problem*

The elastic–plastic response $(\bar{\sigma}_{ij}(t), \bar{\varepsilon}_{ij}(t), \bar{u}_i(t), \bar{\lambda}_\alpha(t))$ of a solid to external actions $(F_i, p_i, \ldots)$ is "contractive" with respect to the Helmholtz free energy, i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega (\bar{\varepsilon}_{ij}^{e^1} - \bar{\varepsilon}_{ij}^{e^2}) D_{ijhk} (\bar{\varepsilon}_{hk}^{e^1} - \bar{\varepsilon}_{hk}^{e^2})\,\mathrm{d}\Omega\right] \leqslant 0 \tag{3.28}$$

where superscripts 1 and 2 denote two different states due to the same history of external actions originating, at time $t = 0$, from different initial states of stresses (see e.g. Simo and Govindjee, 1991). Property eqn (3.28) is valid also for the response of the new approximated time-step problem ΔP (unbarred symbols). In fact, at each time step, using the principle of virtual work and taking into account the semidefinite positiveness of $H_{\alpha\beta}$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega (\varepsilon_{ij}^{e^1} - \varepsilon_{ij}^{e^2}) D_{ijhk} (\varepsilon_{hk}^{e^1} - \varepsilon_{hk}^{e^2})\,\mathrm{d}\Omega\right] = \int_\Omega (\sigma_{ij}^1 - \sigma_{ij}^2)[(\dot{\varepsilon}_{ij}^1 - \dot{\varepsilon}_{ij}^2) - (\dot{\varepsilon}_{ij}^{p^1} - \dot{\varepsilon}_{ij}^{p^2})]\,\mathrm{d}\Omega$$

$$= -\int_\Omega (\sigma_{ij}^1 - \sigma_{ij}^2)\left(\sum_\alpha \dot{\lambda}_\alpha^1 N_{\alpha ij} - \sum_\alpha \dot{\lambda}_\alpha^2 N_{\alpha ij}\right)\mathrm{d}\Omega$$

$$\leqslant \sum_\alpha \int_\Omega \{\dot{\lambda}_\alpha^1[\phi_\alpha(\sigma_{ij}^2) - \phi_\alpha(\sigma_{ij}^1)] + \dot{\lambda}_\alpha^2[\phi_\alpha(\sigma_{ij}^2) - \phi_\alpha(\sigma_{ij}^1)]\}\,\mathrm{d}\Omega. \tag{3.29}$$

However, owing to eqns (3.14)–(3.16), the approximate holonomic time-step problem ΔP must satisfy, for every point $x_i \in \Omega$ and on the entire time interval ΔT, one of the following three conditions:

$$\dot{\lambda}_\alpha = 0 \quad \text{and} \quad \phi_\alpha \leqslant 0 \tag{3.30}$$

$$\phi_\alpha = 0 \quad \text{and} \quad \dot{\lambda}_\alpha \geqslant 0 \tag{3.31}$$

$$\phi_\alpha = 0 \quad \text{and} \quad \dot{\lambda}_\alpha = 0 \tag{3.32}$$

which imply:

$$\dot{\lambda}_\alpha^1 \phi_\alpha(\sigma_{ij}^1) = 0, \quad \dot{\lambda}_\alpha^2 \phi_\alpha(\sigma_{ij}^2) = 0 \tag{3.33}$$

while the vice versa is obviously not true, i.e. eqns (3.33) do not imply eqns (3.14)–(3.16). Then, by substitution of eqns (3.33) into inequality (3.29):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_{\Omega}(\varepsilon_{ij}^{e1}-\varepsilon_{ij}^{e2})D_{ijhk}(\varepsilon_{hk}^{e1}-\varepsilon_{hk}^{e2})\,\mathrm{d}\Omega\right]\leqslant\sum_{\alpha}\int_{\Omega}[\dot{\lambda}_{\alpha}^{1}\phi_{\alpha}(\sigma_{ij}^{2})+\dot{\lambda}_{\alpha}^{2}\phi_{\alpha}(\sigma_{ij}^{1})]\,\mathrm{d}\Omega\leqslant 0 \quad (3.34)$$

because $\dot{\lambda}_\alpha \geqslant 0$ and $\phi_\alpha \leqslant 0$. This proves the contractivity property of the approximate time-step problem ΔP.

4. Extremum principles for holonomic finite-step initial/boundary value problem

Consider the following quadratic functional of the variables $\bar{\sigma}_{ij}^*(x_k, t)$ and $\bar{\lambda}_\alpha^*(x_k, t)$:

$$\Psi[\bar{\sigma}_{ij}^*, \bar{\lambda}_\alpha^*] = -\int_0^T\int_\Omega\sum_\alpha\left(N_{\alpha ij}\bar{\sigma}_{ij}^*-\sum_\beta H_{\alpha\beta}\bar{\lambda}_\beta^*+\phi_\alpha^0\right)\dot{\bar{\lambda}}_\alpha^*\mathrm{d}\Omega\,\mathrm{d}t$$

$$= -\int_0^T\int_\Omega\sum_\alpha\bar{\phi}_\alpha^*\dot{\bar{\lambda}}_\alpha^*\mathrm{d}\Omega\,\mathrm{d}t \quad (4.1)$$

subject to the constraints:

$$\bar{\sigma}_{ij,j}^*+F_i = 0 \quad \text{in } \Omega\times\Delta T \quad (4.2)$$

$$\bar{\sigma}_{ij}^*n_j = p_i \quad \text{on } \Gamma_p\times\Delta T \quad (4.3)$$

$$C_{ijhk}\bar{\sigma}_{hk}^* = \frac{1}{2}(\bar{u}_{i,j}^*+\bar{u}_{j,i}^*)-\sum_\alpha N_{\alpha ij}\bar{\lambda}_\alpha^*-\theta_{ij} \quad \text{in } \Omega\times\Delta T \quad (4.4)$$

$$\bar{u}_i^* = w_i \quad \text{on } \Gamma_u\times\Delta T \quad (4.5)$$

$$\bar{\phi}_\alpha^* = N_{\alpha ij}\bar{\sigma}_{ij}^*-\sum_\beta H_{\alpha\beta}\bar{\lambda}_\beta^*+\phi_\alpha^0 \leqslant 0 \quad \text{in } \Omega\times\Delta T \quad (4.6)$$

$$\dot{\bar{\lambda}}_\alpha^* \geqslant 0 \quad \text{in } \Omega\times\Delta T. \quad (4.7)$$

Statement 1: the fields $\bar{u}_i^*(x_k, t)$, $\bar{\sigma}_{ij}^*(x_k, t)$, $\bar{\lambda}_\alpha^*(x_k, t)$ of displacements, stresses and plastic multipliers are the/a solution of the problem eqns (2.1)–(2.11) if and only if they minimize the functional eqn (4.1) under the constraints eqns (4.2)–(4.7), provided the minimum is zero (otherwise the problem has no solution).

The sign constraints eqns (4.6) and (4.7) imply that:

$$\bar{\Psi} \geqslant 0 \quad (4.8)$$

and that:

$$\bar{\Psi} = 0 \quad \text{if and only if} \quad \bar{\phi}_\alpha^*\dot{\bar{\lambda}}_\alpha^* = 0. \quad (4.9)$$

This proves statement 1. We note that statement 1 is a time-integrated form of the well known statement of maximum plastic dissipation for the original problem $\Delta\bar{P}$, eqns (2.1)–(2.11) (see e.g. Martin, 1975).

It is worth noting that the non-negativeness of the functional eqn (4.1) may be asserted anyway even in the case of lack of normality and/or non-positiveness of the hardening matrix $H_{\alpha\beta}$. The non-negativeness of $\bar{\Psi}$ derives, in fact, only from the sign restrictions on $\bar{\phi}^*_\alpha$ and $\bar{\lambda}^*_\alpha$, i.e. eqns (4.6) and (4.7).

A different form of the functional eqn (4.1), which is more useful for a comparison with previous results, can be obtained by transforming the first addend of the functional eqn (4.1), by virtue of eqn (4.4) and taking into account the identity $\bar{\sigma}^*_{ij}C_{ijhk}\dot{\bar{\sigma}}^*_{hk} = \frac{1}{2}\bar{\sigma}^*_{ij}C_{ijhk}\dot{\bar{\sigma}}^*_{hk} + \frac{1}{2}\bar{\varepsilon}^{e*}_{ij}D_{ijhk}\dot{\bar{\varepsilon}}^{e*}_{hk}$, as follows:

$$-\int_0^T\int_\Omega \sum_\alpha N_{\alpha ij}\bar{\sigma}^*_{ij}\dot{\bar{\lambda}}^*_\alpha \,d\Omega\,dt = \frac{1}{2}\int_0^T\int_\Omega \bar{\sigma}^*_{ij}C_{ijhk}\dot{\bar{\sigma}}^*_{hk}\,d\Omega\,dt + \frac{1}{2}\int_0^T\int_\Omega \bar{\varepsilon}^{e*}_{ij}D_{ijhk}\dot{\bar{\varepsilon}}^{e*}_{hk}\,d\Omega\,dt$$
$$-\int_0^T\int_\Omega \bar{\sigma}^*_{ij}\frac{1}{2}(\dot{\bar{u}}^*_{i,j}-\dot{\bar{u}}^*_{j,i})\,d\Omega\,dt + \int_0^T\int_\Omega \bar{\sigma}^*_{ij}\dot{\theta}_{ij}\,d\Omega\,dt \qquad (4.10)$$

where $\bar{\varepsilon}^{e*}_{ij} = \frac{1}{2}(\bar{u}^*_{i,j}+\bar{u}^*_{j,i}) - \Sigma_\alpha N_{\alpha ij}\bar{\lambda}^*_\alpha - \theta_{ij}$. Using the principle of virtual work, the third term of the second member of eqn (4.10) becomes:

$$-\int_0^T\int_\Omega \bar{\sigma}^*_{ij}\frac{1}{2}(\dot{\bar{u}}^*_{i,j}+\dot{\bar{u}}^*_{j,i})\,d\Omega\,dt$$
$$= -\int_0^T\int_\Omega F_i\dot{\bar{u}}^*_i\,d\Omega\,dt - \int_0^T\int_{\Gamma_p} p_i\dot{\bar{u}}^*_i\,d\Gamma\,dt - \int_0^T\int_{\Gamma_u} \bar{\sigma}^*_{ij}n_j\dot{w}_i\,d\Gamma\,dt. \qquad (4.11)$$

Finally the functional eqn (4.1), using eqns (4.11) and (4.10), becomes:

$$\Psi[\bar{u}^*_i,\bar{\sigma}^*_{ij},\bar{\lambda}^*_\alpha] = \int_0^T\left\{\frac{1}{2}\int_\Omega \bar{\varepsilon}^{e*}_{ij}D_{ijhk}\dot{\bar{\varepsilon}}^{e*}_{hk}\,d\Omega + \frac{1}{2}\int_\Omega \bar{\sigma}^*_{ij}C_{ijhk}\dot{\bar{\sigma}}^*_{hk}\,d\Omega + \sum_{\alpha,\beta}\int_\Omega \bar{\lambda}^*_\alpha H_{\alpha\beta}\dot{\bar{\lambda}}^*_\beta\,d\Omega\right.$$
$$\left.-\int_\Omega F_i\dot{\bar{u}}^*_i\,d\Omega - \int_{\Gamma_u}\bar{\sigma}^*_{ij}n_j\dot{w}_i\,d\Gamma - \int_{\Gamma_p} p_i\dot{\bar{u}}^*_i\,d\Gamma - \sum_\alpha\int_\Omega \phi^0_\alpha\dot{\bar{\lambda}}^*_\alpha(t)\,d\Omega + \int_\Omega \bar{\sigma}^*_{ij}\dot{\theta}_{ij}\,d\Omega\right\}dt. \qquad (4.12)$$

In the case of infinitesimal ΔT (incremental plasticity) and in the case of regular progression of plastic strains (deformation theory), this functional splits into the sum of two other functionals previously introduced by Capurso (1969). Capurso and Maier (1970) and Maier (1969b) (see Appendix).

However, in general, the functional eqn (4.12) cannot be split into the sum of two simpler functionals. This is essentially due to the lack of self-adjointness of the differential operator $d(\cdot)/dt$ with respect to the usual scalar product, that is:

$$\int_0^T a\dot{b}\,dt \neq \int_0^T \dot{a}b\,dt. \qquad (4.13)$$

Making use of the approximate complementarity condition eqn (3.1) it is possible to transform the functional eqn (4.12) as follows:

$$\Psi[u_i^*, \sigma_{ij}^*, \lambda_\alpha^*] = \int_0^T \int_0^T S(t,\tau) \left[\frac{1}{2} \int_\Omega \varepsilon_{ij}^{e*}(t) D_{ijhk} \dot{\varepsilon}_{hk}^{e*}(\tau) \, d\Omega \right.$$

$$+ \frac{1}{2} \int_\Omega \sigma_{ij}^*(t) C_{ijhk} \dot{\sigma}_{hk}^*(\tau) \, d\Omega + \sum_{\alpha,\beta} \int_\Omega \lambda_\alpha^*(t) H_{\alpha\beta} \dot{\lambda}_\beta^*(\tau) \, d\Omega$$

$$- \int_\Omega F_i(t) \dot{u}_i^*(\tau) \, d\Omega - \int_{\Gamma_u} \sigma_{ij}^*(t) n_j \dot{w}_i(\tau) \, d\Gamma$$

$$\left. - \int_{\Gamma_p} p_i(t) \dot{u}_i^*(\tau) \, d\Gamma - \sum_\alpha \int_\Omega \phi_\alpha^0 \dot{\lambda}_\alpha^*(\tau) \, d\Omega + \int_\Omega \sigma_{ij}^*(t) \dot{\theta}_{ij}(\tau) \, d\Omega \right] dt \, d\tau \quad (4.14)$$

under the conditions eqns (4.2)–(4.7) rewritten for the approximate variable fields (i.e. with unbarred symbols). This remark leads to the following:

Statement 2: the fields $u_i^*(x_k, t)$, $\sigma_{ij}^*(x_k, t)$, $\lambda_\alpha^*(x_k, t)$ of displacements, stresses and plastic multipliers are the/a solution of the holonomic problem ΔP (eqns (3.6)–(3.16)) if and only if they minimize the functional eqn (4.14) under the constraints eqns (4.2)–(4.7) (rewritten for the unbarred symbols), provided the minimum is zero (otherwise the problem has no solution).

In fact, the functional eqn (4.14) transforms into:

$$\Psi = -\sum_\alpha \int_0^T \int_\Omega \phi_\alpha^*(t) \int_0^T S(t,\tau) \dot{\lambda}_\alpha^*(\tau) \, d\tau \, d\Omega \, dt \quad (4.15)$$

subject to the conditions:

$$\phi_\alpha^* \leqslant 0 \quad \text{in } \Omega \times \Delta T, \quad \dot{\lambda}_\alpha^* \geqslant 0 \quad \text{in } \Omega \times \Delta T. \quad (4.16)$$

We have, obviously:

$$\Psi \geqslant 0 \quad (4.17)$$

and

$$\Psi = 0 \quad \text{if and only if} \quad \phi_\alpha^*(t) \int_0^T S(t,\tau) \dot{\lambda}_\alpha^*(\tau) \, d\tau = 0 \quad \text{in } \Omega \times \Delta T. \quad (4.18)$$

This proves statement 2.

Differently from functional eqn (4.12), the functional eqn (4.14) can be split into the sum of two simpler functionals Ψ^a and Ψ^b (i.e. $\Psi = \Psi^a + \Psi^b$).

In fact, let us consider the following functionals:

$$\Psi^a[\hat{u}_i, \hat{\lambda}_\alpha] = \int_0^T \int_0^T S(t,\tau) \left[\frac{1}{2} \int_\Omega \hat{\varepsilon}_{ij}^e(t) D_{ijhk} \dot{\hat{\varepsilon}}_{hk}^e(\tau) \, d\Omega + \frac{1}{2} \sum_{\alpha,\beta} \int_\Omega \hat{\lambda}_\alpha(t) H_{\alpha\beta} \dot{\hat{\lambda}}_\beta(\tau) \, d\Omega \right.$$

$$\left. - \int_\Omega F_i(t) \dot{\hat{u}}_i(\tau) \, d\Omega - \int_{\Gamma_p} p_i(t) \dot{\hat{u}}_i(\tau) \, d\Gamma - \sum_\alpha \int_\Omega \phi_\alpha^0 \dot{\hat{\lambda}}_\alpha(\tau) \, d\Omega \right] dt \, d\tau \quad (4.19)$$

where:

$$\hat{\varepsilon}^e_{ij} = \frac{1}{2}(\hat{u}_{i,j}+\hat{u}_{j,i})-\sum_\alpha N_{\alpha ij}\hat{\lambda}_\alpha-\theta_{ij} \tag{4.20}$$

subject to the constraints:

$$\hat{u}_i = w_i \quad \text{on } \Gamma_u \times \Delta T \tag{4.21}$$

$$\hat{\lambda}_\alpha \geqslant 0 \quad \text{in } \Omega \times \Delta T \tag{4.22}$$

and

$$\Psi^b[\sigma'_{ij},\lambda'_\alpha] = \int_0^T\int_0^T S(t,\tau)\left[\frac{1}{2}\int_\Omega \sigma'_{ij}(t)C_{ijhk}\dot{\sigma}'_{hk}(\tau)\,\mathrm{d}\Omega+\frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \lambda'_\alpha(t)H_{\alpha\beta}\dot{\lambda}'_\beta(\tau)\,\mathrm{d}\Omega - \int_{\Gamma_u}\sigma'_{ij}(t)n_j\dot{w}_i(\tau)\,\mathrm{d}\Gamma+\int_\Omega \sigma'_{ij}(t)\dot{\theta}_{ij}(\tau)\,\mathrm{d}\Omega\right]\mathrm{d}t\,\mathrm{d}\tau \tag{4.23}$$

subject to:

$$\sigma'_{ij,j}+F_i = 0 \quad \text{in } \Omega\times\Delta T \tag{4.24}$$

$$\sigma'_{ij}n_j = p_i \quad \text{on } \Gamma_p\times\Delta T \tag{4.25}$$

$$\phi'_\alpha = N_{\alpha ij}\sigma'_{ij}-\sum_\beta H_{\alpha\beta}\lambda'_\beta+\phi^0_\alpha \leqslant 0 \quad \text{in } \Omega\times\Delta T. \tag{4.26}$$

It is possible to verify that $\Psi = \Psi^a+\Psi^b$ and, moreover, the constraints relevant to functional Ψ are the union of the constraints relevant to functionals Ψ^a and Ψ^b.

Statement 3: the fields $\hat{u}_i(x_k,t)$, $\hat{\lambda}_\alpha(x_k,t)$ of displacements and plastic multipliers which are a solution of holonomic problem ΔP (eqns (3.6)–(3.16)) (if a solution exists), make the functional eqn (4.19) minimum under the constraints eqns (4.21)–(4.22), while the fields $\sigma'_{ij}(x_k,t)$, $\lambda'_\alpha(x_k,t)$ of stresses and plastic multipliers, which are a solution of the same holonomic problem, make the functional eqn (4.23) minimum under the constraints eqns (4.24)–(4.26).

Proof. In order to prove the first part of the statement, it suffices to show that the difference

$$\Delta\Psi^a = \Psi^a[\hat{u}_i,\hat{\lambda}_\alpha]-\Psi^a[u_i,\lambda_\alpha] \tag{4.27}$$

is always non-negative for any arbitrary $\hat{u}_i$, $\hat{\lambda}_\alpha$, satisfying conditions eqns (4.21)–(4.22). Assuming:

$$\Delta u_i = \hat{u}_i-u_i \tag{4.28}$$

$$\Delta\lambda_\alpha = \hat{\lambda}_\alpha-\lambda_\alpha \tag{4.29}$$

and integrating by part with respect to the τ variable, the difference eqn (4.27) becomes

$$\Delta\Psi^a = \int_0^T\int_0^T R(t,\tau)\left[\frac{1}{2}\int_\Omega \Delta\varepsilon^e_{ij}(t)D_{ijhk}\,\Delta\varepsilon^e_{hk}(\tau)\,\mathrm{d}\Omega+\frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \Delta\lambda_\alpha(t)H_{\alpha\beta}\,\Delta\lambda_\beta(\tau)\,\mathrm{d}\Omega + \int_\Omega \varepsilon^e_{ij}(t)D_{ijhk}\,\Delta\varepsilon^e_{hk}(\tau)\,\mathrm{d}\Omega+\sum_{\alpha,\beta}\int_\Omega \lambda_\alpha(t)H_{\alpha\beta}\,\Delta\lambda_\beta(\tau)\,\mathrm{d}\Omega - \int_\Omega F_i(t)\,\Delta u_i(\tau)\,\mathrm{d}\Omega-\int_{\Gamma_p}p_i(t)\,\Delta u_i(\tau)\,\mathrm{d}\Gamma-\sum_\alpha\int_\Omega \phi^0_\alpha\,\Delta\lambda_\alpha(\tau)\,\mathrm{d}\Omega\right]\mathrm{d}\tau\,\mathrm{d}t. \tag{4.30}$$

Taking into account the eqns (2.5), (2.6), (2.8) and (4.20), the third integral of the r.h.s. may be transformed as follows:

$$\int_{\Omega} \varepsilon^e_{ij}(t) D_{ijhk}\, \Delta\varepsilon^e_{hk}(\tau)\, \mathrm{d}\Omega = \int_{\Omega} \sigma_{ij}(t) \left[\Delta\varepsilon_{ij}(\tau) - \sum_{\alpha} N_{\alpha ij}\, \Delta\lambda_{\alpha}(\tau) \right] \mathrm{d}\Omega \tag{4.31}$$

and, for the Gauss lemma, it becomes:

$$\int_{\Omega} (\sigma_{ij}(t)\, \Delta u_i(\tau))_{,j}\, \mathrm{d}\Omega - \int_{\Omega} \sigma_{ij,j}(t)\, \Delta u_i(\tau)\, \mathrm{d}\Omega - \int_{\Omega} \sigma_{ij}(t) \sum_{\alpha} N_{\alpha ij}\, \Delta\lambda_{\alpha}(\tau)\, \mathrm{d}\Omega$$
$$= \int_{\Gamma_p} \sigma_{ij}(t) n_j\, \Delta u_i(\tau)\, \mathrm{d}\Gamma - \int_{\Omega} \sigma_{ij,j}(t)\, \Delta u_i(\tau)\, \mathrm{d}\Omega - \int_{\Omega} \sigma_{ij}(t) \sum_{\alpha} N_{\alpha ij}\, \Delta\lambda_{\alpha}(\tau)\, \mathrm{d}\Omega. \tag{4.32}$$

Therefore, noting that the equilibrium equations are satisfied at the solution, we obtain:

$$\Delta\Psi^a = \int_0^T \int_0^T R(t,\tau) \left[\frac{1}{2} \int_{\Omega} \Delta\varepsilon^e_{ij}(t) D_{ijhk}\, \Delta\varepsilon^e_{hk}(\tau)\, \mathrm{d}\Omega + \frac{1}{2} \sum_{\alpha,\beta} \int_{\Omega} \Delta\lambda_{\alpha}(t) H_{\alpha\beta}\, \Delta\lambda_{\beta}(\tau)\, \mathrm{d}\Omega \right] \mathrm{d}\tau\, \mathrm{d}t$$
$$- \int_0^T \int_0^T R(t,\tau) \int_{\Omega} \sum_{\alpha} \left[N_{\alpha ij}\sigma_{ij}(t) - \sum_{\beta} H_{\alpha\beta}\lambda_{\beta}(t) + \phi^0_{\alpha} \right] \Delta\lambda_{\alpha}(\tau)\, \mathrm{d}\Omega\, \mathrm{d}\tau\, \mathrm{d}t. \tag{4.33}$$

Owing to eqn (3.4), the first integral of $\Delta\Psi^a$ is always non-negative. The second integral, after integration by part with respect to the variable τ, can be rewritten as follows:

$$-\sum_{\alpha} \int_0^T \int_{\Omega} \phi_{\alpha}(t) \int_0^T S(t,\tau)(\hat{\dot{\lambda}}_{\alpha} - \dot{\lambda}_{\alpha})\, \mathrm{d}\tau\, \mathrm{d}\Omega\, \mathrm{d}t. \tag{4.34}$$

By using relation eqn (3.1), every term of the sum eqn (4.34) reduces to

$$-\int_0^T \int_{\Omega} \phi_{\alpha}(t) \int_0^T S(t,\tau) \hat{\dot{\lambda}}_{\alpha}\, \mathrm{d}\tau\, \mathrm{d}\Omega\, \mathrm{d}t \tag{4.35}$$

which, because of eqns (3.14), (3.3) and (4.22), is always non-negative and equal to zero if and only if

$$\int_0^T S(t,\tau) \hat{\dot{\lambda}}_{\alpha}(\tau)\, \mathrm{d}\tau = 0 \quad \text{where } \phi_{\alpha} < 0. \tag{4.36}$$

This proves that

$$\Psi^a[\hat{u}_i, \hat{\lambda}_{\alpha}] \geqslant \Psi^a[u_i, \lambda_{\alpha}] \tag{4.37}$$

for any field $\hat{u}_i$, and $\hat{\lambda}_{\alpha}$ satisfying the conditions eqns (4.21)–(4.22), the equality sign holding if

$$\hat{u}_i = u_i \quad \text{in } \Omega \times \Delta T \tag{4.38}$$

$$\hat{\lambda}_\alpha = \lambda_\alpha \quad \text{in } \Omega \times \Delta T. \tag{4.39}$$

Similarly, the proof of the second part of Statement 3 amounts to show that the difference

$$\Delta\Psi^b = \Psi^b[\sigma'_{ij}, \lambda'_\alpha] - \Psi^b[\sigma_{ij}, \lambda_\alpha] \tag{4.40}$$

is always non-negative for any arbitrary σ'_{ij}, λ'_α satisfying conditions eqns (4.24)–(4.26). Assuming:

$$\Delta\sigma_{ij} = \sigma'_{ij} - \sigma_{ij} \tag{4.41}$$

$$\Delta\lambda_\alpha = \lambda'_\alpha - \lambda_\alpha \tag{4.42}$$

the difference eqn (4.40) becomes

$$\begin{aligned}\Delta\Psi^b = \int_0^T \int_0^T R(t,\tau) \Bigg[& \frac{1}{2} \int_\Omega \Delta\sigma_{ij}(t) C_{ijhk} \, \Delta\sigma_{hk}(\tau) \, d\Omega \\ & + \frac{1}{2} \sum_{\alpha,\beta} \int_\Omega \Delta\lambda_\alpha(t) H_{\alpha\beta} \, \Delta\lambda_\beta(\tau) \, d\tau \\ & + \int_\Omega \sigma_{ij}(t) C_{ijhk} \, \Delta\sigma_{hk}(\tau) \, d\Omega + \sum_{\alpha,\beta} \int_\Omega \lambda_\alpha(t) H_{\alpha\beta} \, \Delta\lambda_\beta(\tau) \, d\Omega \\ & - \int_{\Gamma_u} \Delta\sigma_{ij}(t) n_j w_i(\tau) \, d\Gamma + \int_\Omega \Delta\sigma_{ij}(t) \theta_{ij}(\tau) \, d\Omega \Bigg] d\tau \, dt. \end{aligned} \tag{4.43}$$

Using Gauss lemma, the difference $\Delta\Psi^b$ reduces to:

$$\begin{aligned}\Delta\Psi^b = & \int_0^T \int_0^T R(t,\tau) \left[\frac{1}{2} \int_\Omega \Delta\sigma_{ij}(t) C_{ijhk} \, \Delta\sigma_{hk}(\tau) \, d\Omega + \frac{1}{2} \sum_{\alpha,\beta} \int_\Omega \Delta\lambda_\alpha(t) H_{\alpha\beta} \, \Delta\lambda_\beta(\tau) \, d\Omega \right] d\tau \, dt \\ & - \int_0^T \int_0^T R(t,\tau) \int_\Omega \sum_\alpha \left[N_{\alpha ij} \, \Delta\sigma_{ij}(t) - \sum_\beta H_{\alpha\beta} \, \Delta\lambda_\beta(t) \right] \lambda_\alpha(\tau) \, d\Omega \, d\tau \, dt. \end{aligned} \tag{4.44}$$

Owing to eqn (3.4), the first integral of $\Delta\Psi^b$ is always non-negative. The second integral, after integration by part with respect to the variable τ, can be rewritten as follows:

$$-\sum_\alpha \int_0^T \int_\Omega (\phi'_\alpha(t) - \phi_\alpha(t)) \int_0^T S(t,\tau) \dot{\lambda}_\alpha(\tau) \, d\tau \, d\Omega \, dt \tag{4.45}$$

and, by virtue of relation eqn (3.1), every term of the sum eqn (4.45) reduces to

$$-\int_0^T \int_\Omega \phi'_\alpha(t) \int_0^T S(t,\tau) \dot{\lambda}_\alpha(\tau) \, d\tau \, d\Omega \, dt \tag{4.46}$$

which, because of eqns (4.26), (3.15) and (3.3), is always non-negative and equal to zero if

$$\phi'_\alpha(t) = 0 \quad \text{where} \quad \int_0^T S(t,\tau)\dot{\lambda}_\alpha(\tau)\,d\tau > 0. \tag{4.47}$$

This proves that

$$\Psi^b[\sigma'_{ij}, \lambda'_\alpha] \geqslant \Psi^b[\sigma_{ij}, \lambda_\alpha] \tag{4.48}$$

for any field σ'_{ij} and λ'_α satisfying the conditions eqns (4.24)–(4.26), the equality sign holding if

$$\sigma'_{ij} = \sigma_{ij} \quad \text{in } \Omega \times \Delta T \tag{4.49}$$

$$\lambda'_\alpha = \lambda_\alpha \quad \text{in} \quad \Omega \times \Delta T. \tag{4.50}$$

5. Time discretization and links with some classical integration rules

For the sake of simplicity, only the discretization of the functional $\Psi^a[\hat{u}_i, \hat{\lambda}_\alpha]$ of Statement 3 is here considered.

In order to easily show the connection of the preceding results with some classical time integration rules, among the numerous discretization schemes for the numerical solution of variational problems, the following usual space-time independent interpolations of the unknown fields $\hat{u}_i$, $\hat{\lambda}_\alpha$ is adopted:

$$\hat{u}_i = \tilde{\mathbf{N}}^u(t)\mathbf{U}_i(\mathbf{x}) \tag{5.1}$$

$$\hat{\lambda}_\alpha = \tilde{\mathbf{N}}^\lambda(t)\mathbf{\Lambda}_\alpha(\mathbf{x}) \tag{5.2}$$

where tilde means transposition. Vectors $\mathbf{N}^u(t)$ and $\mathbf{N}^\lambda(t)$ collect suitable interpolation time functions, while $\mathbf{U}_i(\mathbf{x})$ and $\mathbf{\Lambda}_\alpha(\mathbf{x})$ represent vectors of unknown parameters. By substitution of eqns (5.1) and (5.2) into the functional eqn (4.19) we have:

$$\begin{aligned}
\Psi^a[\mathbf{U}_i, \mathbf{\Lambda}_\alpha] = {} & \frac{1}{2}\int_\Omega \tilde{\mathbf{E}}_{ij}\mathbf{M}^u D_{ijhk}\mathbf{E}_{hk}\,d\Omega + \frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \tilde{\mathbf{\Lambda}}_\alpha \mathbf{M}^\lambda \mathbf{\Lambda}_\beta N_{\alpha ij} D_{ijhk} N_{\beta hk}\,d\Omega \\
& - \sum_\alpha \int_\Omega \tilde{\mathbf{E}}_{ij}\mathbf{M}^{u\lambda} D_{ijhk} N_{\alpha hk}\mathbf{\Lambda}_\alpha\,d\Omega + \frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \tilde{\mathbf{\Lambda}}_\alpha \mathbf{M}^\lambda H_{\alpha\beta}\mathbf{\Lambda}_\beta\,d\Omega \\
& - \int_\Omega \tilde{\mathbf{U}}_i \mathbf{m}_i^F\,d\Omega - \int_{\Gamma_p} \tilde{\mathbf{U}}_i \mathbf{m}_i^p\,d\Gamma - \sum_\alpha \int_\Omega \phi_\alpha^0 \tilde{\mathbf{\Lambda}}_\alpha \mathbf{m}^\lambda\,d\Omega \\
& - \int_\Omega \tilde{\mathbf{E}}_{ij}\mathbf{m}_{ij}^{u0}\,d\Omega + \sum_\alpha \int_\Omega \tilde{\mathbf{\Lambda}}_\alpha N_{\alpha ij}\mathbf{m}_{ij}^{\lambda 0}\,d\Omega
\end{aligned} \tag{5.3}$$

where

$$\mathbf{M}^u = \int_0^T\int_0^T R(t,\tau)\mathbf{N}^u(t)\tilde{\mathbf{N}}^u(\tau)\,dt\,d\tau \tag{5.4}$$

$$\mathbf{M}^{\lambda} = \int_0^T \int_0^T R(t,\tau)\mathbf{N}^{\lambda}(t)\tilde{\mathbf{N}}^{\lambda}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.5}$$

$$\mathbf{M}^{u\lambda} = \int_0^T \int_0^T R(t,\tau)\mathbf{N}^{u}(t)\tilde{\mathbf{N}}^{\lambda}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.6}$$

$$\mathbf{m}_i^F = \int_0^T \int_0^T R(t,\tau)F_i(t)\mathbf{N}^{u}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.7}$$

$$\mathbf{m}_i^p = \int_0^T \int_0^T R(t,\tau)p_i(t)\mathbf{N}^{u}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.8}$$

$$\mathbf{m}_{ij}^{\lambda\theta} = \int_0^T \int_0^T R(t,\tau)D_{ijhk}\theta_{hk}(t)\mathbf{N}^{\lambda}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.9}$$

$$\mathbf{m}_{ij}^{u\theta} = \int_0^T \int_0^T R(t,\tau)D_{ijhk}\theta_{hk}(t)\mathbf{N}^{u}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.10}$$

$$\mathbf{m}_i^{\lambda} = \int_0^T \int_0^T R(t,\tau)\mathbf{N}^{\lambda}(\tau)\,\mathrm{d}t\,\mathrm{d}\tau \tag{5.11}$$

and

$$\mathbf{E}_{ij} = \tfrac{1}{2}(\mathbf{U}_{i,j}+\mathbf{U}_{j,i}). \tag{5.12}$$

It is easy to show that for a suitable choice of the weight function $S(t,\tau)$ and for an appropriate time approximation of the unknowns, some well known time integration schemes are recovered. In particular if we assume that the unknowns are linear functions of time t (i.e. $\mathbf{N}^u(t) = t/T$, $\mathbf{N}^{\lambda}(t) = t/T$, $\mathbf{U}_i = U_i$, $\mathbf{\Lambda}_\alpha = \Lambda_\alpha$, where U_i and Λ_α are the values of the unknowns at the end of the time step $[0, T]$), eqns (5.1) and (5.2) assume the following form:

$$\hat{u}_i(t) = \frac{t}{T}U_i \tag{5.13}$$

$$\hat{\lambda}_\alpha(t) = \frac{t}{T}\Lambda_\alpha. \tag{5.14}$$

Moreover, assuming that the prescribed displacements $w_i(t)$ are a linear function of time in $[0, T]$ and that $|\lim_{t\to 0}(\theta_i(t)/t)| < \infty$, the functional eqn (5.3) transforms into the following:

$$\Psi^a[U_i,\Lambda_\alpha] = \int_0^T \int_0^T R(t,\tau)\Bigg[\left(\frac{1}{2}\int_\Omega E^e_{ij}D_{ijhk}E^e_{hk}\,\mathrm{d}\Omega + \frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \Lambda_\alpha H_{\alpha\beta}\Lambda_\beta\,\mathrm{d}\Omega\right) \times\frac{t\tau}{T^2} + \left(\sum_\alpha \int_\Omega \phi_\alpha(0)\Lambda_\alpha\,\mathrm{d}\Omega - \int_\Omega F_i(t)U_i\,\mathrm{d}\Omega - \int_{\Gamma_p} p_i(t)U_i\,\mathrm{d}\Gamma\right)\frac{\tau}{T}\Bigg]\mathrm{d}t\,\mathrm{d}\tau \tag{5.15}$$

where

$$E^e_{ij}(t) = \frac{1}{2}(U_{i,j}+U_{j,i}) - \sum_\alpha N_{\alpha ij}\Lambda_\alpha - \theta_{ij}(t)\frac{T}{t} \tag{5.16}$$

subject to the constraints

$$U_i = w_i(T) \quad \text{on } \Gamma_u; \quad \Lambda_\alpha \geqslant 0. \tag{5.17}$$

The optimality conditions for the discretized problem are (where α, $\beta = 1, 2, \ldots, m$)

$$\int_0^T \int_0^T R(t,\tau)\left[\Xi_{ij,j}(t)\frac{t\tau}{T^2} + F_i(t)\frac{\tau}{T}\right]\mathrm{d}\tau\,\mathrm{d}t = 0 \quad \text{in } \Omega \tag{5.18}$$

$$\int_0^T \int_0^T R(t,\tau)\left[\Xi_{ij}(t)n_j\frac{t\tau}{T^2} - p_i(t)\frac{\tau}{T}\right]\mathrm{d}\tau\,\mathrm{d}t = 0 \quad \text{on } \Gamma_p \tag{5.19}$$

$$\frac{1}{2}(U_{i,j}+U_{j,i}) = E^e_{ij}(t) + \sum_\alpha N_{\alpha ij}\Lambda_\alpha + \theta_{ij}(t)\frac{T}{t} \quad \text{in } \Omega \tag{5.20}$$

$$U_i = w_i(T) \quad \text{on } \Gamma_u \tag{5.21}$$

$$\Xi_{ij}(t) = D_{ijhk}E^e_{hk}(t) \tag{5.22}$$

$$\Phi_\alpha(t) = N_{\alpha ij}\left(\sigma^0_{ij} + \Xi_{ij}(t)\frac{t}{T}\right) - \sum_\beta H_{\alpha\beta}\left(\lambda^0_\beta + \Lambda_\beta\frac{t}{T}\right) - r_\alpha \tag{5.23}$$

$$\int_0^T \int_0^T R(t,\tau)\Phi_\alpha(t)\frac{\tau}{T}\mathrm{d}\tau\,\mathrm{d}t \leqslant 0 \tag{5.24}$$

$$\Lambda_\alpha \geqslant 0 \tag{5.25}$$

$$\Lambda_\alpha \int_0^T \int_0^T R(t,\tau)\Phi_\alpha(t)\frac{\tau}{T}\mathrm{d}\tau\,\mathrm{d}t = 0. \tag{5.26}$$

For $R(t,\tau) = \delta(t-T)\delta(\tau-T)$ [$\delta(t)$ being the "Dirac delta function"], the set of previous eqns (5.18)–(5.26) becomes:

$$\Xi_{ij,j}(T) + F_i(T) = 0 \quad \text{in } \Omega \tag{5.27}$$

$$\Xi_{ij}(T)n_j - p_i(T) = 0 \quad \text{on } \Gamma_p \tag{5.28}$$

$$\frac{1}{2}(U_{i,j}+U_{j,i}) = E^e_{ij}(T) + \sum_\alpha N_{\alpha ij}\Lambda_\alpha + \theta_{ij}(T) \quad \text{in } \Omega \tag{5.29}$$

$$U_i = w_i(T) \quad \text{on } \Gamma_u \tag{5.30}$$

$$\Xi_{ij}(T) = D_{ijhk}E^e_{hk}(T) \tag{5.31}$$

$$\Phi_\alpha(T) = N_{\alpha ij}(\sigma^0_{ij} + \Xi_{ij}(T)) - \sum_\beta H_{\alpha\beta}(\lambda^0_\beta + \Lambda_\beta) - r_\alpha \tag{5.32}$$

$$\Phi_\alpha(T) \leqslant 0 \tag{5.33}$$

$$\Lambda_\alpha \geqslant 0 \tag{5.34}$$

$$\Lambda_\alpha \Phi_\alpha(T) = 0 \tag{5.35}$$

which can be generated by the backward difference method, while, for $R(t,\tau) = \delta(t-(T/2))\delta(\tau-(T/2))$, the set of eqns (5.18)–(5.26) becomes:

$$\frac{1}{2}\Xi_{ij,j}\left(\frac{T}{2}\right)+F_i\left(\frac{T}{2}\right) = 0 \quad \text{in } \Omega \tag{5.36}$$

$$\frac{1}{2}\Xi_{ij}\left(\frac{T}{2}\right)n_j - p_i\left(\frac{T}{2}\right) = 0 \quad \text{on } \Gamma_p \tag{5.37}$$

$$\frac{1}{2}(U_{i,j}+U_{j,i}) = E^e_{ij}\left(\frac{T}{2}\right)+\sum_\alpha N_{\alpha ij}\Lambda_\alpha + 2\theta_{ij}\left(\frac{T}{2}\right) \quad \text{in } \Omega \tag{5.38}$$

$$U_i = w_i(T) \quad \text{on } \Gamma_u \tag{5.39}$$

$$\Xi_{ij}\left(\frac{T}{2}\right) = D_{ijhk}E^e_{hk}\left(\frac{T}{2}\right) \tag{5.40}$$

$$\Phi_\alpha\left(\frac{T}{2}\right) = N_{\alpha ij}\left(\sigma^0_{ij}+\frac{1}{2}\Xi_{ij}\left(\frac{T}{2}\right)\right)-\sum_\beta H_{\alpha\beta}\left(\lambda^0_\beta+\frac{1}{2}\Lambda_\beta\right)-r_\alpha \tag{5.41}$$

$$\Phi_\alpha\left(\frac{T}{2}\right) \leqslant 0 \tag{5.42}$$

$$\Lambda_\alpha \geqslant 0 \tag{5.43}$$

$$\Lambda_\alpha \Phi_\alpha\left(\frac{T}{2}\right) = 0. \tag{5.44}$$

If $\theta_{ij}(t)$ is a linear function of time in $[0, T]$ [i.e. $\theta_{ij}(t) = (t/T)\theta_{ij}(T)$], the following holds:

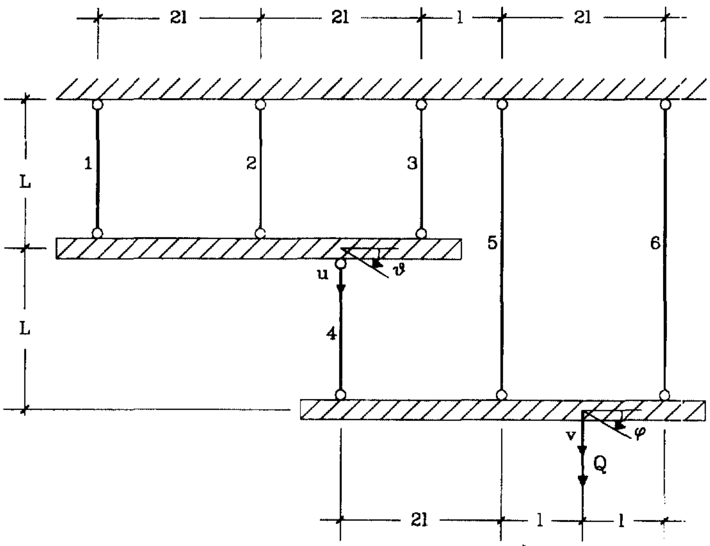
$$E^e_{ij}\left(\frac{T}{2}\right) \equiv E^e_{ij}(T), \quad \Xi_{ij}\left(\frac{T}{2}\right) \equiv \Xi_{ij}(T) \tag{5.45}$$

and then the last set of equations, substituting in it identities eqn (5.45), corresponds to the set which can be generated by the trapezoidal rule.

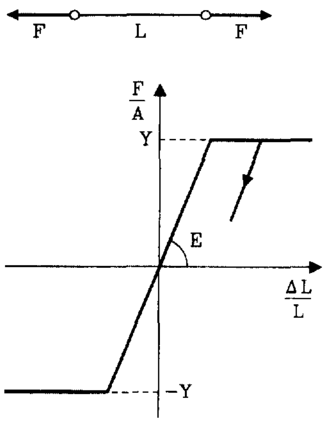
6. Illustrative example

A simple example of the use of the function $\Psi^a[\hat{u}_i, \hat{\lambda}_\alpha]$, eqn (4.19), is presented in order to evidence the easiness of using multiple degrees of freedom time discretization schemes for each time step

ΔT. The well known Hodge's six-bar truss problem of Fig. 3 was considered (Hodge, 1973). Two solutions were found, the first one, using (for the unknown field u_i and λ_α) the linear shape function $f_0(t)$ (see Fig. 4) as indicated in eqns (5.13) and (5.14); the second one, using the following quadratic functions [see eqns (5.1) and (5.2)]:



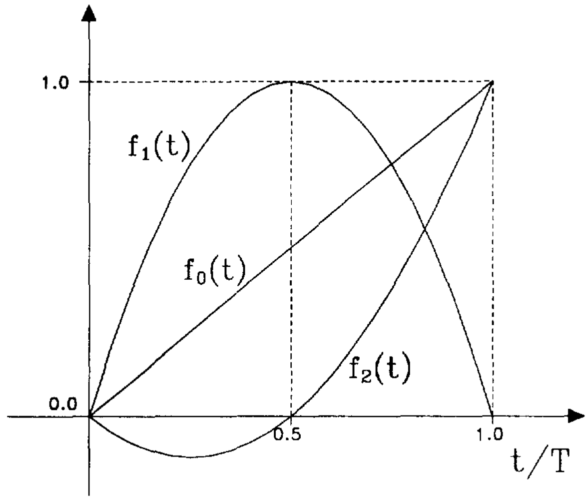
(a)



Bar	Yield Stress
1	Y
2	12 Y
3	8 Y
4	20 Y
5	185 Y
6	130 Y

(b)

Fig. 3. Hodge's six-bar truss considered for the numerical application of the proposed approximate formulation; (a) geometry and loading, (b) mechanical properties of the vertical bars; Young's modulus E and cross section area A are assumed equal for all bars.



$$f_0(t)=\frac{t}{T} \qquad f_1(t)=\frac{4}{T^2}(tT-t^2) \qquad f_2(t)=\frac{1}{T^2}(2t^2-Tt)$$

Fig. 4. Linear and quadratic shape functions $f_0(t), f_1(t), f_2(t)$ adopted in the numerical example.

$$\hat{u}_i(t)=[f_1(t)\quad f_2(t)]\begin{bmatrix}U_{i1}\\ U_{i2}\end{bmatrix}=\tilde{\mathbf{N}}^u(t)\mathbf{U}_i(\mathbf{x}) \tag{6.1}$$

$$\hat{\lambda}_\alpha(t)=[f_1(t)\quad f_2(t)]\begin{bmatrix}\Lambda_{\alpha 1}\\ \Lambda_{\alpha 2}\end{bmatrix}=\tilde{\mathbf{N}}^\lambda(t)\mathbf{\Lambda}_\alpha(\mathbf{x}) \tag{6.2}$$

where the shape functions $f_1(t)$ and $f_2(t)$ are given in Fig. 4, being U_{i1}, $\Lambda_{\alpha 1}$ and U_{i2}, $\Lambda_{\alpha 2}$ the displacements and the plastic multipliers at the mid point (1) and at the end (2) of the time step interval, respectively. For both the linear and quadratic time interpolation schemes, the same weight function $S_3(t,\tau)$ [eqn (3.21)] was used and the problem was reduced to the minimization of a quadratic function under a set of linear inequality constraints.

For each approximation scheme, several solutions were found for different numbers (8, 12 and 16) of time discretization steps of the time interval $[t^*, T_f]$ where t^* is the time of the first appearance of the yielding in a bar (see Fig. 5). In Fig. 5 a comparison is shown between the exact and the approximate solutions, while in Figs 6 and 7 percentage errors of displacements, rotations and stresses are evaluated as a function of loading history and for different numbers of time steps. Of course, when the exact solution vanishes, the percentage error is no longer significant. For this reason, in Fig. 7 the results relevant to bars 1, 2, 3 and 4 are omitted; in fact, in those cases the exact stress plots change in sign one or more times.

As foreseeable, better results are obtained by the analysis performed with a quadratic interpolation function.

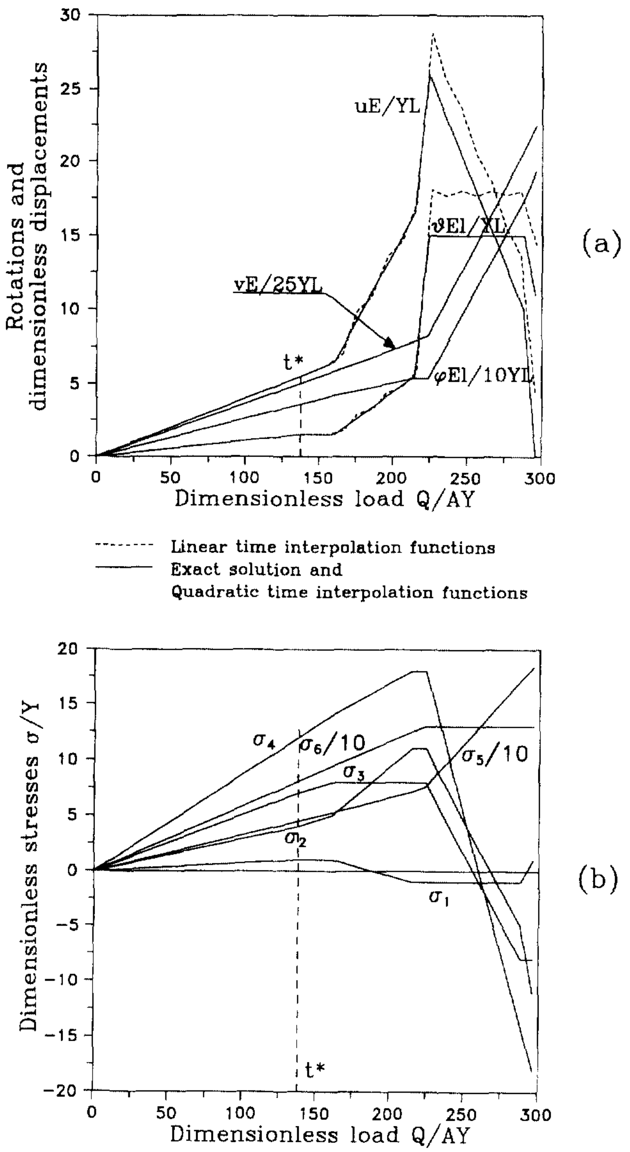


Fig. 5. Hodge's exact solution (solid lines), in terms of displacements (a) and of stresses (b), compared to the approximate solutions (dashed lines) obtained through linear and quadratic time-interpolation functions with a fixed number (16) of time steps. Both the linear and quadratic approximate solutions practically coincide with the exact stress response (b), while some differences occur in the displacements for high value of Q, in the case of the linear time-approximation only (a).

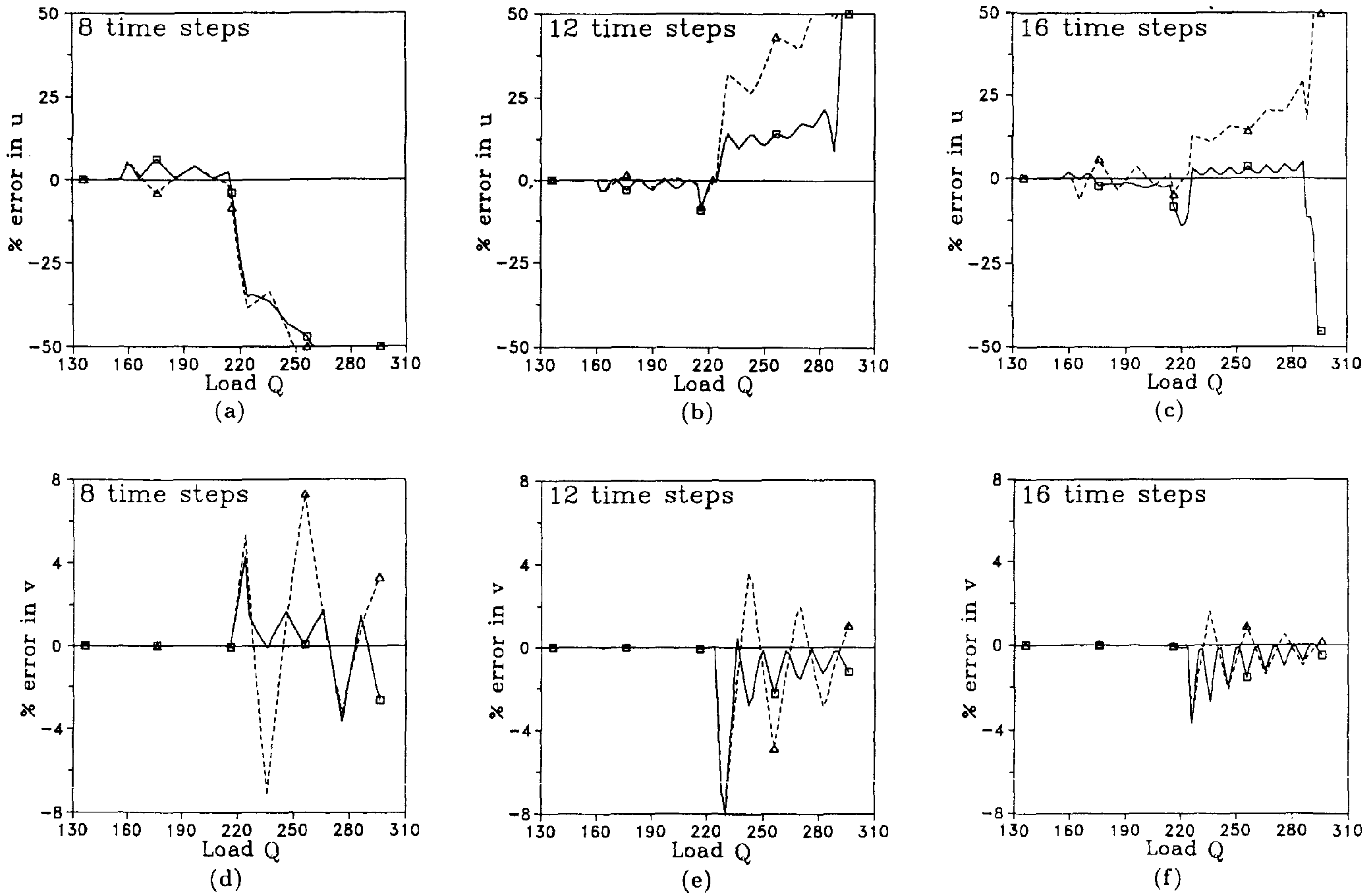
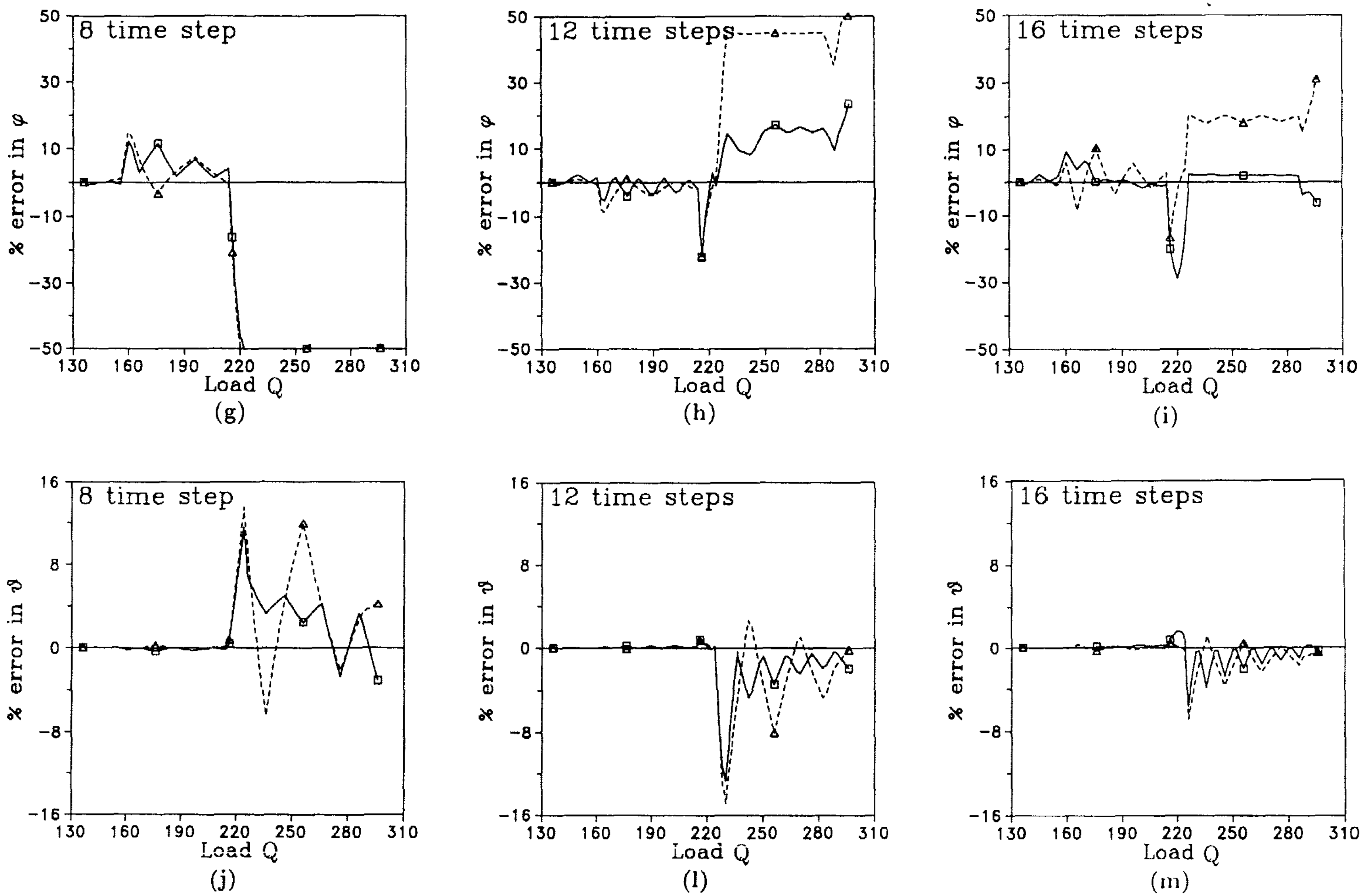


Fig. 6. Percentage error, for increasing values of load Q and for different numbers of time steps (8, 12, 16), of displacements $u(Q)$, $v(Q)$ [(a)–(f)] and rotations $\varphi(Q)$, $\theta(Q)$ [(g)–(n)] of the two horizontal rigid bars of Fig. 3 using linear and quadratic time interpolation functions [i.e. the percentage error of $u(Q)$ is $100\ (u(Q)-u_{exact}(Q))/u_{exact}(Q)$].



△- - -△ Linear time approximation
□——□ Quadratic time approximation

Fig. 6.—*Continued.*

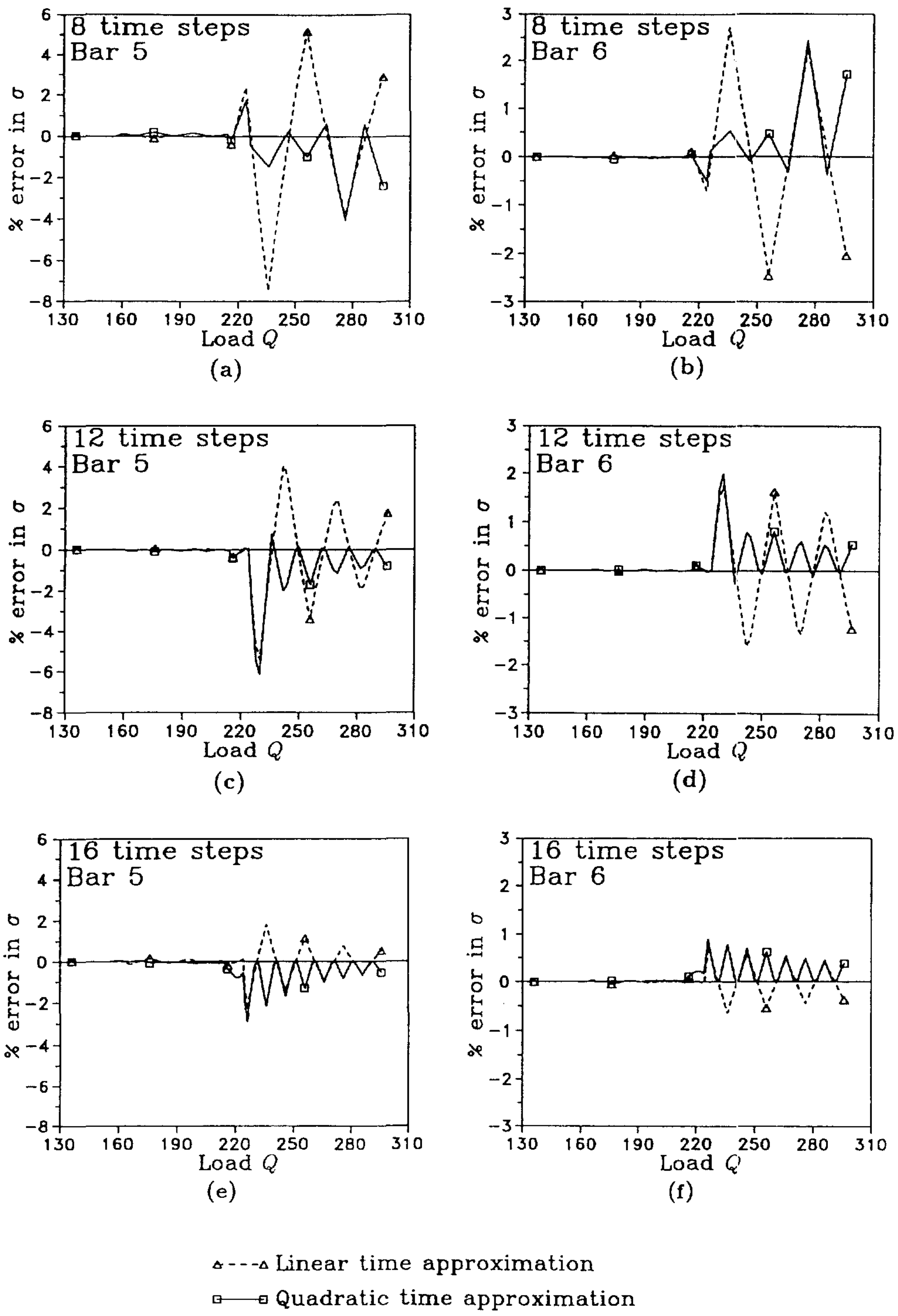


Fig. 7. Percentage error, for increasing values of load Q and for different numbers of time steps (8, 12, 16), of stresses σ in bars 5 and 6, using linear and quadratic time interpolation functions [i.e. $100\ (\sigma(Q)-\sigma_{exact}(Q))/\sigma_{exact}(Q)$].

It is worth noting that by using higher order interpolation functions it will be possible to capture as accurately as desired any non-linear behaviour deriving from the nonlinearity of the hardening, of the yield surface and of the elastic part of the constitutive law. However, capturing of the non-linear behaviour due to the non-holonomic character of the constitutive law, is related to the amplitude of the time step considered when the unloading processes occur inside the time step, so the choice of the higher order interpolation functions has no effect.

Furthermore, it is worth noting that, in the time step where unloading occurs, the percentage error of the stress state associated to a vanishing value of the plastic multiplier $\dot{\lambda}_\alpha$ (corresponding to regions where $\dot{\lambda}_\alpha \equiv 0$) may become large independently of the order of the interpolation function assumed, as the real value of the yield function $\bar{\phi}_\alpha$ may be largely negative, while, in the numerical computation, $\phi_\alpha = 0$.

Finally, in Fig. 8 some comparisons among the results of the proposed method with those of classical techniques, are shown. Figures 8(a) and 8(b) show the numerical results in terms of vertical displacement u and rotation φ, respectively. The results concerning the vertical displacement v and the rotation θ are omitted, because all the approximate methods give results very close to the exact solution. Figure 8(c) shows the comparisons in terms of stress in bar 1. The analogous plots for the other bars are omitted because of the substantial agreement of all the approximate solutions with the exact one. It appears that the proposed method gives results in good accordance with those of the trapezoidal rule, but in any case with smaller oscillations. This seems to be a consequence of the choice of the weight function. In fact, the adopted weight function R_3 [see Fig. 2(f)] takes larger values around $t = \tau = T/2$ and in some way simulates the behaviour of the trapezoidal rule which (as shown in the previous section) has a "Dirac delta function" at $t = \tau = T/2$ as weight function.

7. Concluding remarks

For an elastic–plastic material with piecewise-linearized yield surface, linear hardening and associated constitutive law, in the context of small strains and displacements, the following results have been achieved:

(1) An extremal formulation was developed for the finite single time step solution of the elastic–plastic problem, for each given loading process, taking into account the fully non-holonomic material behaviour (local-unloading included).
(2) It has been shown that the above functional specializes to the sum of other functionals already introduced by Capurso and by Maier, if an infinitesimal time step or a finite time step with a holonomic constitutive law are considered.
(3) Through a suitable change of the complementarity condition the elastic–plastic problem has been transformed into an approximate holonomic version. The elastic–plastic response of the new holonomic problem is contractive with respect to the Helmholtz free energy.
(4) Two extremal formulations of the new holonomic problem have been established, which characterize the complete evolution of the body response in a prescribed time interval.
(5) Finally, by the adoption of shape functions over the whole time interval it has been shown

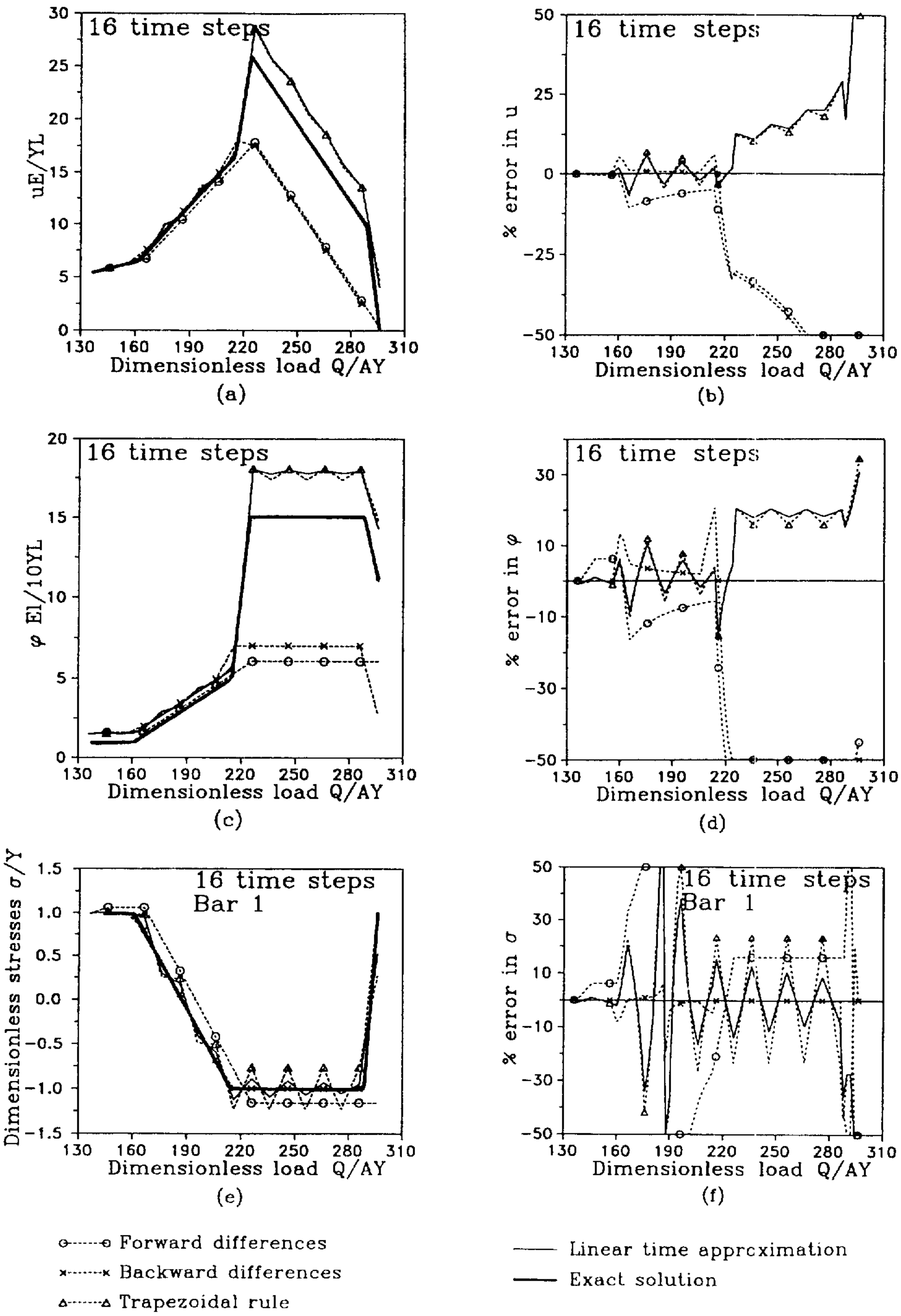


Fig. 8. Comparison between the exact solution of the problem of Fig. 3 and the numerical solutions obtained by the proposed method (using linear time approximation) and by other known methods (forward difference, backward differences and trapezoidal rule); (a), (b) comparison in terms of displacements; (c), (d) comparison in terms of rotations; (e), (f) comparison in terms of stresses.

how well-known integration schemes for the solution of the original problem are equivalent to the equations governing the new holonomic problem.

The following remarks are worth making:

(a) In contrast to classical formulations, the present one allows for the discretization of the elastic–plastic problem both in time and space, particularly with Ritz-type discretization in time.
(b) The choice of the weight function $S(t, \tau)$ allows one to define a priori the idealization adopted for the real problem and then, when the solution of the corresponding new holonomic formulation is found, it is possible to know the idealization of the real problem that the solution corresponds to. In general, this correspondence is not known, owing to the difficulty of determining the consequences of the time discretization schemes adopted in order to find a numerical solution.
(c) A major advantage of the possibility to choose the approximate schemes of the real behaviour (using suitable weight functions $S(t, \tau)$) is that the choice can be made in such a way to have an a priori guarantee of important solution properties such as existence, uniqueness and stability that are generally not so easy to establish.
(d) Finally, it is expected that the proposed formulation may be extended to more general constitutive laws, as in the case of non-linear hardening and non-linearized yield surface. This topic, together with the study of the influence of the choice of the weight function $S(t, \tau)$ on the numerical results, may be the object of further investigations.

Acknowledgements

Informal discussions with Professors O. De Donato and G. Maier are gratefully acknowledged. The authors wish to thank one of the reviewers for his useful and accurate remarks and suggestions.

Appendix

Link with known incremental theorems

Consider an infinitesimal increment of external actions $\mathrm{d}F_i$, $\mathrm{d}p_i$, $\mathrm{d}w_i$, $\mathrm{d}\theta_{ij}$ in a time step $\mathrm{d}t$, starting at the initial time t_0 from a known state F_i^0, p_i^0, w_i^0, θ_{ij}^0, σ_{ij}^0, ε_{ij}^0 and λ_α^0. The increasing quantities can be written as:

$$\begin{aligned} \bar{u}_i^* &= u_i^0 + \mathrm{d}\bar{u}_i^* \\ \bar{\varepsilon}_{ij}^* &= \varepsilon_{ij}^0 + \mathrm{d}\bar{\varepsilon}_{ij}^* \\ \bar{\sigma}_{ij}^* &= \sigma_{ij}^0 + \mathrm{d}\bar{\sigma}_{ij}^* \\ \bar{\lambda}_\alpha^* &= \lambda_\alpha^0 + \mathrm{d}\bar{\lambda}_\alpha^* \end{aligned} \tag{A1}$$

Dropping an unessential constant, the functional eqn (4.12) becomes:

$$\bar{\Psi}[\bar{u}_i^*, \bar{\sigma}_{ij}^*, \bar{\lambda}_\alpha^*] = \int_{t_0}^{t_0+\mathrm{d}t} \left[\frac{1}{2} \int_\Omega (\sigma_{ij}^0 + \mathrm{d}\bar{\sigma}_{ij}^*) C_{ijhk} \dot{\bar{\sigma}}_{hk}^* \,\mathrm{d}\Omega \right.$$
$$+ \frac{1}{2} \int_\Omega (\varepsilon_{ij}^{e0} - \mathrm{d}\bar{\varepsilon}_{ij}^{e*}) D_{ijhk} \dot{\bar{\varepsilon}}_{hk}^{e*} \,\mathrm{d}\Omega + \sum_{\alpha,\beta} \int_\Omega (\lambda_\alpha^0 - \mathrm{d}\bar{\lambda}_\alpha^*) H_{\alpha\beta} \dot{\bar{\lambda}}_\beta^* \,\mathrm{d}\Omega \qquad \text{(A2)}$$
$$- \int_\Omega (F_i^0 + \mathrm{d}F_i) \dot{\bar{u}}_i^* \mathrm{d}\Omega - \int_{\Gamma_u} (\sigma_{ij}^0 + \mathrm{d}\bar{\sigma}_{ij}^*) n_j \dot{w}_i \,\mathrm{d}\Gamma$$
$$\left. - \int_{\Gamma_p} (p_i^0 - \mathrm{d}p_i) \dot{\bar{u}}_i^* \mathrm{d}\Gamma + \sum_\alpha \int_\Omega r_\alpha \dot{\bar{\lambda}}_\alpha^* \mathrm{d}\Omega + \int_\Omega (\sigma_{ij}^0 + \mathrm{d}\bar{\sigma}_{ij}^*) \dot{\theta}_{ij} \,\mathrm{d}\Omega \right] \mathrm{d}t.$$

In eqn (A2) all time integrals (in consideration of infinitesimal time step) and some unessential constants can be removed. Then:

$$\bar{\Psi}[\mathrm{d}\bar{u}_i^*, \mathrm{d}\bar{\sigma}_{ij}^*, \mathrm{d}\bar{\lambda}_\alpha^*] = \frac{1}{2} \int_\Omega \mathrm{d}\bar{\sigma}_{ij}^* C_{ijhk} \,\mathrm{d}\bar{\sigma}_{hk}^* \,\mathrm{d}\Omega + \frac{1}{2} \int_\Omega \mathrm{d}\bar{\varepsilon}_{ij}^{e*} D_{ijhk} \,\mathrm{d}\bar{\varepsilon}_{hk}^{e*} \,\mathrm{d}\Omega$$
$$+ \sum_{\alpha,\beta} \int_\Omega \mathrm{d}\bar{\lambda}_\alpha^* H_{\alpha\beta} \,\mathrm{d}\bar{\lambda}_\beta^* \mathrm{d}\Omega - \int_\Omega \mathrm{d}F_i \,\mathrm{d}\bar{u}_i^* \mathrm{d}\Omega - \int_{\Gamma_p} \mathrm{d}p_i \,\mathrm{d}\bar{u}_i^* \mathrm{d}\Gamma \qquad \text{(A3)}$$
$$- \int_{\Gamma_u} \mathrm{d}\bar{\sigma}_{ij}^* n_j \,\mathrm{d}w_i \,\mathrm{d}\Gamma + \int_\Omega \mathrm{d}\bar{\sigma}_{ij}^* \,\mathrm{d}\theta_{ij} \,\mathrm{d}\Omega - \sum_\alpha \int_\Omega \phi_\alpha^0 \,\mathrm{d}\bar{\lambda}_\alpha^* \mathrm{d}\Omega$$

where we write:

$$\phi_\alpha^0 = \phi_\alpha(0) = N_{\alpha ij} \sigma_{ij}^0 - r_\alpha - \sum_\beta H_{\alpha\beta} \lambda_\beta^0. \qquad \text{(A4)}$$

Let Ω_p denote the region of Ω in which $\phi_\alpha^0 = 0$ and Ω_e the remaining part of Ω. Now we write functional (A2) in the form:

$$\bar{\Psi}[\mathrm{d}\bar{u}_i^*, \mathrm{d}\bar{\sigma}_{ij}^*, \mathrm{d}\bar{\lambda}_\alpha^*] = \bar{\Psi}_1[\mathrm{d}\bar{u}_i^*, \mathrm{d}\bar{\sigma}_{ij}^*, \mathrm{d}\bar{\lambda}_\alpha^*] + \bar{\Psi}_2[\mathrm{d}\bar{\lambda}_\alpha^*] \qquad \text{(A5)}$$

where, taking into account that $\phi_\alpha^0 = 0$ in Ω_p:

$$\bar{\Psi}_1[\mathrm{d}\bar{u}_i^*, \mathrm{d}\bar{\sigma}_{ij}^*, \mathrm{d}\bar{\lambda}_\alpha^*] = \frac{1}{2} \int_\Omega \mathrm{d}\bar{\sigma}_{ij}^* C_{ijhk} \,\mathrm{d}\bar{\sigma}_{hk}^* \,\mathrm{d}\Omega + \frac{1}{2} \int_\Omega \mathrm{d}\bar{\varepsilon}_{ij}^{e*} D_{ijhk} \,\mathrm{d}\bar{\varepsilon}_{hk}^{e*} \,\mathrm{d}\Omega$$
$$+ \sum_{\alpha,\beta} \int_{\Omega_p} \mathrm{d}\bar{\lambda}_\alpha^* H_{\alpha\beta} \,\mathrm{d}\bar{\lambda}_\beta^* \mathrm{d}\Omega - \int_\Omega \mathrm{d}F_i \,\mathrm{d}\bar{u}_i^* \mathrm{d}\Omega - \int_{\Gamma_p} \mathrm{d}p_i \,\mathrm{d}\bar{u}_i^* \mathrm{d}\Gamma \qquad \text{(A6)}$$
$$- \int_{\Gamma_u} \mathrm{d}\bar{\sigma}_{ij}^* n_j \,\mathrm{d}w_i \,\mathrm{d}\Gamma + \int_\Omega \mathrm{d}\bar{\sigma}_{ij}^* \,\mathrm{d}\theta_{ij} \,\mathrm{d}\Omega$$

$$\bar{\Psi}_2[\mathrm{d}\bar{\lambda}_\alpha^*] = \sum_{\alpha,\beta} \int_{\Omega_e} \mathrm{d}\bar{\lambda}_\alpha^* H_{\alpha\beta} \,\mathrm{d}\bar{\lambda}_\beta^* \mathrm{d}\Omega - \sum_\alpha \int_{\Omega_e} \phi_\alpha^0 \,\mathrm{d}\bar{\lambda}_\alpha^* \mathrm{d}\Omega. \qquad \text{(A7)}$$

By substituting eqn (A1) into the linear constraints eqns (4.2)–(4.7) we obtain:

$$\mathrm{d}\bar{\sigma}^*_{ij,j}+\mathrm{d}F_i=0 \quad \text{in } \Omega \tag{A8}$$

$$\mathrm{d}\bar{\sigma}^*_{ij}n_j=\mathrm{d}p_i \quad \text{on } \Gamma_p \tag{A9}$$

$$\mathrm{d}\bar{\varepsilon}^{e*}_{ij}=\frac{1}{2}(\mathrm{d}\bar{u}^*_{i,j}+\mathrm{d}\bar{u}^*_{j,i})-\sum_\alpha N_{\alpha ij}\,\mathrm{d}\bar{\lambda}^*_\alpha-\mathrm{d}\theta_{ij} \quad \text{in } \Omega \tag{A10}$$

$$\mathrm{d}\bar{u}^*_i=\mathrm{d}w_i \quad \text{on } \Gamma_u \tag{A11}$$

$$\mathrm{d}\bar{\phi}^*_\alpha\leqslant 0 \quad \text{in } \Omega_p \tag{A12}$$

$$\mathrm{d}\bar{\lambda}^*_\alpha\geqslant 0 \quad \text{in } \Omega_p \tag{A13}$$

$$\phi^0_\alpha+\mathrm{d}\bar{\phi}^*_\alpha\leqslant 0 \quad \text{in } \Omega_e \tag{A14}$$

$$\mathrm{d}\bar{\lambda}^*_\alpha\geqslant 0 \quad \text{in } \Omega_e. \tag{A15}$$

In the elastic zone Ω_e the constraint eqn (A14) is certainly satisfied for every $\mathrm{d}\bar{\phi}^*_\alpha$ since it is infinitesimal with respect to $\bar{\phi}^*_\alpha$; then it can be omitted. The problem:

$$\min_{\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha}\{\bar{\Psi}=\bar{\Psi}[\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha] \quad \text{subject to (A8)–(A15)}\} \tag{A16}$$

can be regarded as:

$$\min_{\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha}\{\bar{\Psi}_1=\bar{\Psi}_1[\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha] \quad \text{subject to (A8)–(A15)}\}$$
$$+\min_{\mathrm{d}\bar{\lambda}^*_\alpha}\{\bar{\Psi}_2=\bar{\Psi}_2[\mathrm{d}\bar{\lambda}^*_\alpha] \quad \text{subject to (A15)}\}. \tag{A17}$$

In fact, either the unknown functions of $\bar{\Psi}_1$ are not present in $\bar{\Psi}_2$ and vice versa, or, if they are present in both the functionals, they pertain to domains with a vanishing intersection. The solution of the minimum problem:

$$\min_{\mathrm{d}\bar{\lambda}^*_\alpha}\{\bar{\Psi}_2=\bar{\Psi}_2[\mathrm{d}\bar{\lambda}^*_\alpha] \quad \text{subject to (A15)}\} \tag{A18}$$

is obviously:

$$\mathrm{d}\bar{\lambda}^*_\alpha=0 \quad \text{in } \Omega_e. \tag{A19}$$

Therefore, the original minimum problem can be replaced by the following:

$$\min_{\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha}\{\bar{\Psi}_1=\bar{\Psi}_1[\mathrm{d}\bar{u}^*_i,\mathrm{d}\bar{\sigma}^*_{ij},\mathrm{d}\bar{\lambda}^*_\alpha] \quad \text{subject to (A8)–(A13)}\}. \tag{A20}$$

Functional eqn (A6), derived from eqn (4.12) for an infinitesimal time-step, subject to the constraints eqn (A8)–(A13), is equivalent to the sum of previous functionals introduced by Capurso (1969) and Capurso and Maier (1970).

Link with known holonomic, finite time-interval theorems

By integration by part, using the principle of virtual work and the relation eqn (4.4), the functional eqn (4.12) becomes:

$$\bar{\Psi}[\bar{u}_i^*, \bar{\varepsilon}_{ij}^{e*}, \bar{\sigma}_{ij}^*, \bar{\lambda}_\alpha^*] = \frac{1}{2}\int_\Omega \bar{\sigma}_{ij}^*(T) C_{ijhk} \bar{\sigma}_{hk}^*(T)\,\mathrm{d}\Omega + \frac{1}{2}\int_\Omega \bar{\varepsilon}_{ij}^{e*}(T) D_{ijhk} \bar{\varepsilon}_{hk}^{e*}(T)\,\mathrm{d}\Omega$$

$$+\frac{1}{2}\sum_{\alpha,\beta}\int_\Omega \bar{\lambda}_\alpha^*(T) H_{\alpha\beta} \bar{\lambda}_\beta^*(T)\,\mathrm{d}\Omega + \sum_\alpha \int_0^T \int_\Omega \dot{\bar{\sigma}}_{ij}^* N_{\alpha ij} \bar{\lambda}_\alpha^*\,\mathrm{d}\Omega\,\mathrm{d}t$$

$$-\int_\Omega F_i(T)\bar{u}_i^*(T)\,\mathrm{d}\Omega - \int_{\Gamma_u} \bar{\sigma}_{ij}^*(T) n_j w_i(T)\,\mathrm{d}\Gamma - \int_{\Gamma_p} p_i(T)\bar{u}_i^*(T)\,\mathrm{d}\Gamma$$

$$+\sum_\alpha \int_\Omega r_\alpha \bar{\lambda}_\alpha^*(T)\,\mathrm{d}\Omega + \int_\Omega \bar{\sigma}_{ij}^*(T)\theta_{ij}(T)\,\mathrm{d}\Omega. \tag{A21}$$

Consider the particular case of regular progression of plastic strains ($\bar{\lambda}_\alpha^*$ monotonously increasing). i.e. a loading history which never causes elastic unloading. As a consequence, $\bar{\lambda}_\alpha^* = 0$ everywhere until plastic deformations appear for the first time (at $t \leqslant \bar{t}$); whereas for any $t > \bar{t}$ we have $\bar{\phi}_\alpha^* = 0$. Therefore the following must hold:

$$\dot{\bar{\sigma}}_{ij}^* N_{\alpha ij} \bar{\lambda}_\alpha^* = 0 \tag{A22}$$

since $\bar{\lambda}_\alpha^* = 0$ for $t \leqslant \bar{t}$, while $\bar{\phi}_\alpha^* = 0$ for $t > \bar{t}$ from which $\dot{\bar{\phi}}_\alpha^* = 0$, i.e.:

$$\dot{\bar{\sigma}}_{ij}^* N_{\alpha ij} = H_{\alpha\beta} \dot{\bar{\lambda}}_\beta^*. \tag{A23}$$

As a consequence of eqns (A22) and (A23) the unknown functions $\bar{u}_i^*(x_k, t)$, $\bar{\sigma}_{ij}^*(x_k, t)$, $\bar{\lambda}_\alpha^*(x_k, t)$, depend only on the final time T. Then the functional eqn (A21) becomes:

$$\bar{\Psi}[\bar{u}_i^*(T), \bar{\varepsilon}_{ij}^{e*}(T), \bar{\sigma}_{ij}^*(T), \bar{\lambda}_\alpha^*(T)] = \frac{1}{2}\int_\Omega \bar{\sigma}_{ij}^*(T) C_{ijhk} \bar{\sigma}_{hk}^*(T)\,\mathrm{d}\Omega$$

$$+\frac{1}{2}\int_\Omega \bar{\varepsilon}_{ij}^{e*}(T) D_{ijhk} \bar{\varepsilon}_{hk}^{e*}(T)\,\mathrm{d}\Omega$$

$$+\sum_{\alpha,\beta}\int_\Omega \bar{\lambda}_\alpha^*(T) H_{\alpha\beta} \bar{\lambda}_\beta^*(T)\,\mathrm{d}\Omega - \int_\Omega F_i(T)\bar{u}_i^*(T)\,\mathrm{d}\Omega - \int_{\Gamma_u} \bar{\sigma}_{ij}^*(T) n_j w_i(T)\,\mathrm{d}\Gamma$$

$$-\int_{\Gamma_p} p_i(T)\bar{u}_i^*(T)\,\mathrm{d}\Gamma + \sum_\alpha \int_\Omega r_\alpha \bar{\lambda}_\alpha^*(T)\,\mathrm{d}\Omega + \int_\Omega \bar{\sigma}_{ij}^*(T)\theta_{ij}(T)\,\mathrm{d}\Omega. \tag{A24}$$

As a consequence of the increasing monotony of $\bar{\lambda}_\alpha^*$ and of (A22), (A23), being $\bar{\Psi}$ now only dependent on T values, the constraints eqns (4.2)–(4.7) become (the condition $\dot{\bar{\lambda}}_\alpha^* \geqslant 0$ being changed with $\bar{\lambda}_\alpha^* \geqslant 0$):

$$\bar{\sigma}_{ij,j}^*(T) + F_i(T) = 0 \quad \text{in } \Omega \tag{A25}$$

$$\bar{\sigma}_{ij}^*(T) n_j = p_i(T) \quad \text{on } \Gamma_p \tag{A26}$$

$$\bar{\varepsilon}_{ij}^{e*}(T) = \frac{1}{2}(\bar{u}_{i,j}^*(T) + \bar{u}_{j,i}^*(T)) - \sum_\alpha N_{\alpha ij}\bar{\lambda}_\alpha^*(T) - \theta_{ij}(T) \quad \text{in } \Omega \tag{A27}$$

$$\bar{u}_i^*(T) = w_i(T) \quad \text{on } \Gamma_u \tag{A28}$$

$$\bar{\phi}_\alpha^*(T) = N_{\alpha ij}\bar{\sigma}_{ij}^*(T) - \sum_\beta H_{\alpha\beta}\bar{\lambda}_\beta^*(T) - r_\alpha \leqslant 0 \quad \text{in } \Omega \tag{A29}$$

$$\bar{\lambda}_\alpha^*(T) \geqslant 0 \quad \text{in } \Omega. \tag{A30}$$

Functional eqn (A24), derived from eqn (4.12) on the assumption of a finite holonomic time step and subject to the constraints eqns (A25)–(A30), is equivalent to the sum of two other functionals previously introduced by Maier (1969b).

References

Borino, G., Fuschi, P. and Polizzotto, C. (1989) Consistent time modelling for the evolutive analysis of elastic–plastic solids. In *Computational Plasticity II*, ed. D. R. J. Owen, E. Hinton and E. Onate, Vol. I, p. 85. Pineridge Press, Swansea.

Capurso, M. (1969) Principi di minimo per la soluzione incrementale dei problemi elastoplastici. Nota I–II, *Rend. Acc. Naz. Lincei.*

Capurso, M. and Maier, G. (1970) Incremental elastoplastic analysis and quadratic optimization, *Meccanica*, no. 2, Vol. V.

De Donato, O. (1974) On piecewise linear constitutive laws in plasticity, Technical Report No. 39 I.S.T.C. Politecnico di Milano.

De Donato, O. and Maier, G. (1972) Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity. *Int. J. Num. Methods Engng* **4**, 307–329.

De Donato, O. and Maier, G. (1973) Finite element elastoplastic analysis by quadratic programming: the multistage method. *2nd International Conference on Structural Mechanics in Reactor Technology*, Berlin, 10–14 September.

Franchi, A. and Genna, F. (1984) Minimum principles and initial stress method in elastic–plastic analysis. *Eng. Struct.* **6**, 65–69.

Hill, R. (1957) A general theory of uniqueness and stability in elastic–plastic solids. *J. Mech. Phys. Solids* **5**, 229.

Hodge, P. G. (1973) Complete solution for elastic–plastic trusses. *SIAM J. Appl. Math.* **25**, 435–447.

Hodge, P. G. (1976) Piecewise linear strainhardening plasticity. In *Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects*, ed. J. A. Strieklin and K. J. Saezalski. Applied Mechanics Symposia Series, ASME, AMD, Vol. 20, New York.

Maier, G. (1969a) Complementary plastic work theorems in piecewise-linear elastoplasticity. *International Journal of Solids and Structures* **5**, 261–270.

Maier, G. (1969b) Teoremi di minimo in termini finiti per continui elastoplastici con leggi costitutive linearizzate a tratti. *Rendiconti dell'Istituto Lombardo di Scienze e Lettere* **103**, 1066.

Maier, G. (1970a) A matrix structural theory of piecewise-linear plasticity with interacting yield planes. *Meccanica* **5**(1), 54–66.

Maier, G. (1970b) A minimum principle for incremental elastoplasticity with non associated flow laws. *Journal of the Mechanics and Physics of Solids* **18**, 319.

Martin, J. B (1975) *Plasticity: Fundamentals and General Results*. MIT Press, Cambridge, MA.

Martin, J. B. (1989) Integration along the path of loading in elastic–plastic problems. In *Computational Plasticity II*, ed. D. R. J. Owen, E. Hinton and E. Onate. Vol. I, pp. 1–15. Pineridge Press, Swansea.

Ortiz, M. and Martin, J. B. (1989) Symmetry preserving return mapping algorithms and incrementally extremal paths: a unification of concepts. *Int. J. Num. Meth. in Engng* **28**, 1839–1853.

Ortiz, M. and Popov, E. P. (1985) Accuracy and stability of integration algorithms for elasto-plastic constitutive relations. *Int. J. Num. Meth. in Engng* **21**, 1561–1576.

Polizzotto, C. and Zito, M. (1990) A variational formulation of the BE for elastic–plastic analysis. In *Discretization Methods in Structural Mechanics*, ed. G. Kuhn and H. Mang, p. 201. Springer Verlag, Berlin.

Ponter, A. R. S. and Martin, J. B. (1972) Some extremal properties and energy theorems for inelastic materials and their relationship to the deformation theory of plasticity. *Journal of the Mechanics and Physics of Solids* **20,** 281–300.
Reddy, B. D., Martin, J. B. and Griffin, T. B. (1986) Extremal paths and holonomic constitutive laws in elastoplasticity. *UCT/CSIR, Appl. Mech. Res. Unit*, Technical Report, No. 82, University of Cape Town.
Simo, J. C. and Govindjee, S. (1991) Non-linear B-stability and symmetry preserving return mapping algorithms for plasticity and viscoplasticity. *Int. J. Num. Meth. Engng* **31,** 151–176.
Simo, J. C., Kennedy, J. G. and Govindjee, S. (1988) Non-smooth multisurface plasticity and viscoplasticity loading/unloading conditions and numerical algorithms. *Int. J. Num. Meth. Engng* **26,** 2161–2185.
Simo, J. C. and Taylor, R. C. (1985) Consistent tangent operator for rate-independent elasto-plasticity. *Comp. Meth. in Appl. Mech. and Engng* **48,** 101–118.